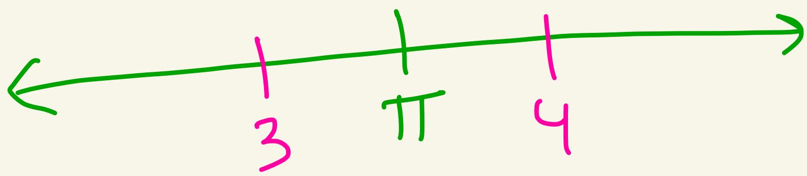
Math 4650
9/3/25

Lemma: Let $a > 0$ be a real number.
Then there exists a natural number $m \in \mathbb{N}$ with
$$m - 1 \le a < m$$

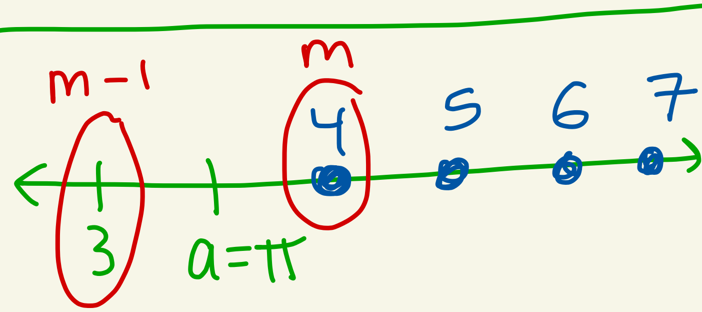
Ex: $a = \pi \approx 3.14\ldots$

$m = 4$



proof: Let
$$E = \{ n \mid n \in \mathbb{N} \text{ and } a < n \}$$

By the Archimedean property, $E \neq \emptyset$.
Let m be the smallest element of E, which



$E = \{4, 5, 6, 7, \ldots\}$
$a = \pi$

We know exists because $E \subseteq \mathbb{N}$.

Then, $m-1 \notin E$.

Then either $m-1 \notin \mathbb{N}$ or $m-1 \leq a$.

<u>case 1:</u> Suppose $m-1 \notin \mathbb{N}$.

Then, $m=1$ and $m-1=0$.

Since $m \in E$ we know $a<m$.

Since $0<a$ we know $m-1<a$.

Thus, $m-1<a<m$.

<u>case 2:</u> Suppose $m-1 \leq a$.

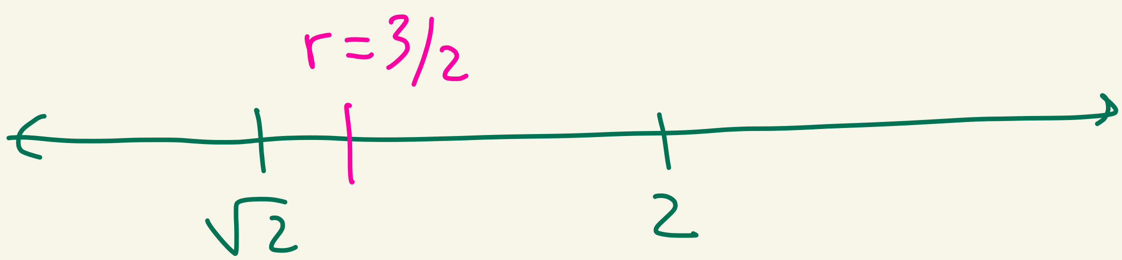
Since $m \in E$ we know $a<m$.

Then, $m-1 \leq a < m$. $\blacksquare$

Theorem ($\mathbb{Q}$ is dense in $\mathbb{R}$)

Given $x, y \in \mathbb{R}$ with $x < y$, there exists $r \in \mathbb{Q}$ with $x < r < y$.

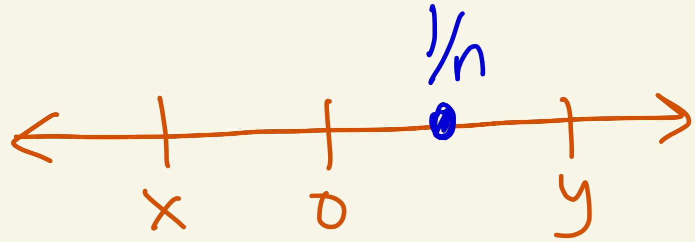
Ex: $x = \sqrt{2} \approx 1.414\ldots$, $y = 2$



Proof:

Case 1: Suppose $x \leq 0 < y$.

Pick a natural number n with $n > 1/y$



Then, $0 < \frac{1}{n} < y$.

Let $r = 1/n$.

Then, $x \le 0 < r < y$.

<u>case 2:</u> Suppose $0 < x < y$.

Then, $y - x > 0$.

Pick a natural number n with $n > 1/(y-x)$.

So, $\frac{1}{n} < y - x$.

Then, $1 < ny - nx$.

Thus, $\boxed{nx + 1 < ny.}$

By the previous lemma there exists $m \in \mathbb{N}$ with

$$\boxed{m - 1 \le nx < m.}$$

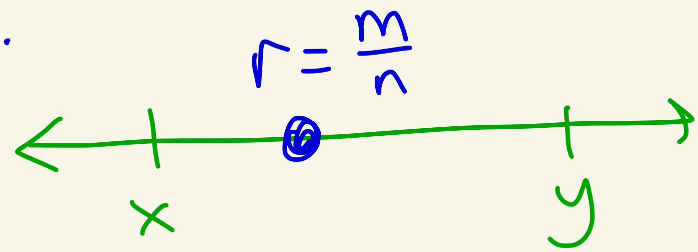
ok to use lemma since $n > 0$, $x > 0$ so $nx > 0$

Thus, $m \le nx + 1$.

Then, $m \leq nx + 1 < ny$.

In summary, $nx < m < ny$.

So, $x < \frac{m}{n} < y$.



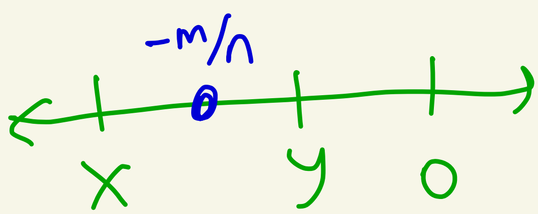
Let $r = \frac{m}{n}$.

<u>Case 3:</u> Suppose $x < y < 0$.

Then, $0 < -y < -x$.

By case 2 there exists $\frac{m}{n} \in \mathbb{Q}$ with $-y < \frac{m}{n} < -x$.

Then, $x < -\frac{m}{n} < y$.

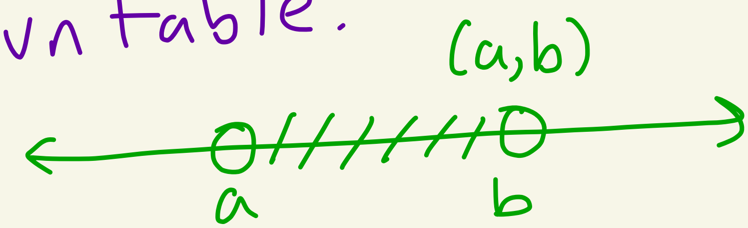


Let $r = -\frac{m}{n}$. $\blacksquare$

<u>Theorem:</u> Let $a, b \in \mathbb{R}$ with $a < b$. Then there exists an irrational number x with $a < x < b$.

<u>proof:</u>

From math 3450, the interval (a,b) is uncountable.



Since $\mathbb{Q}$ are countable we know $\mathbb{Q} \cap (a,b)$ is countable because $\mathbb{Q} \cap (a,b) \subseteq \mathbb{Q}$.

Thus, $\underbrace{(a,b) - \left(\mathbb{Q} \cap (a,b)\right)}_{\text{same as: } (a,b) \setminus \left(\mathbb{Q} \cap (a,b)\right)} \neq \emptyset.$

Let $x \in (a,b) - (\mathbb{Q} \cap (a,b))$.
Then x is irrational
and $a < x < b$. $\blacksquare$

Topic 2 - Sequences

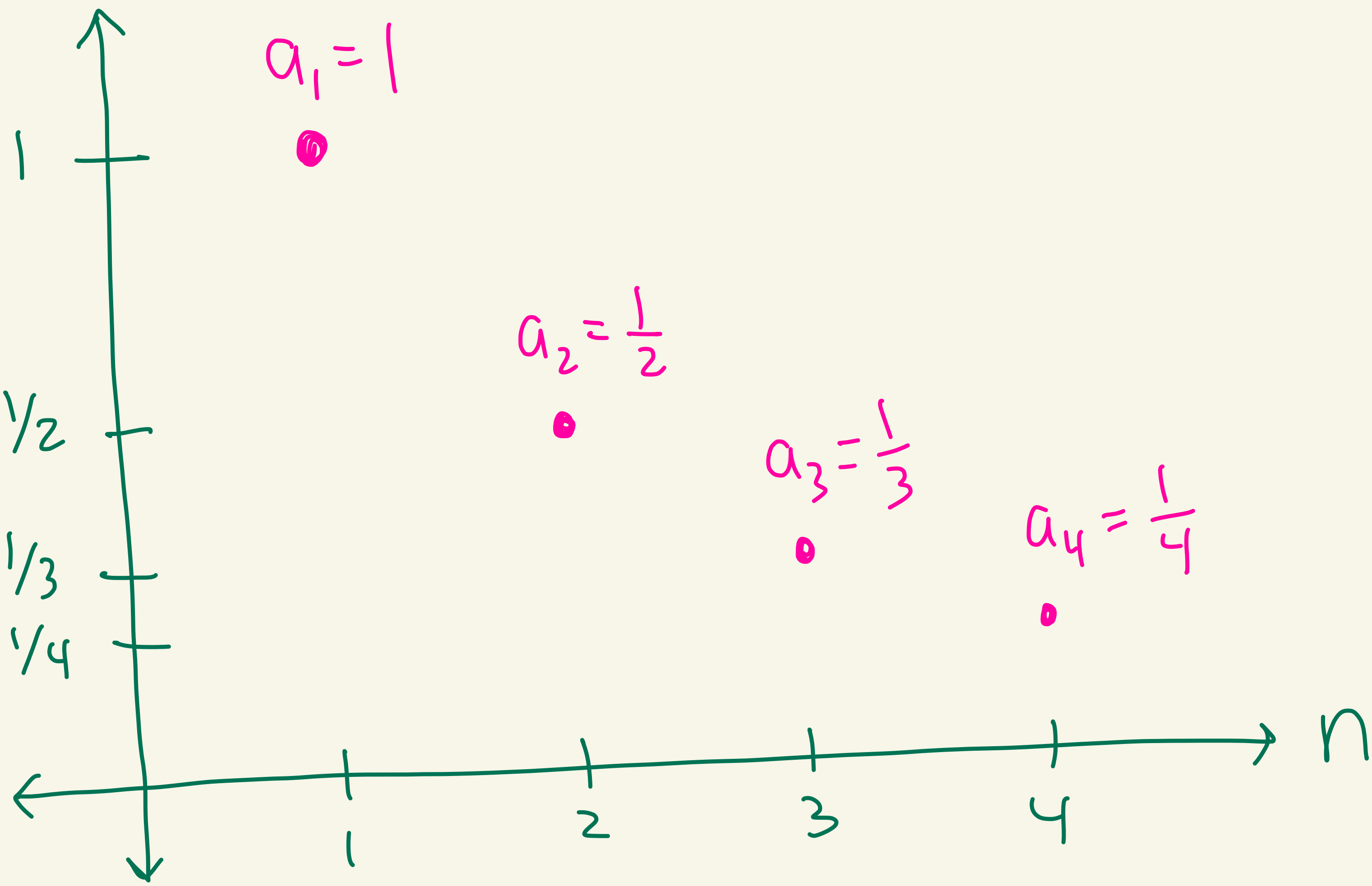
<u>Def:</u> A <u>sequence</u> of real numbers written (a_n) or $\underbrace{(a_n)_{n=1}^{\infty}}_{\text{can start at } n>1 \text{ also}}$ is an ordered list of real numbers:

$$a_1, a_2, a_3, a_4, a_5, a_6, \ldots$$

Can start at another number other than 1

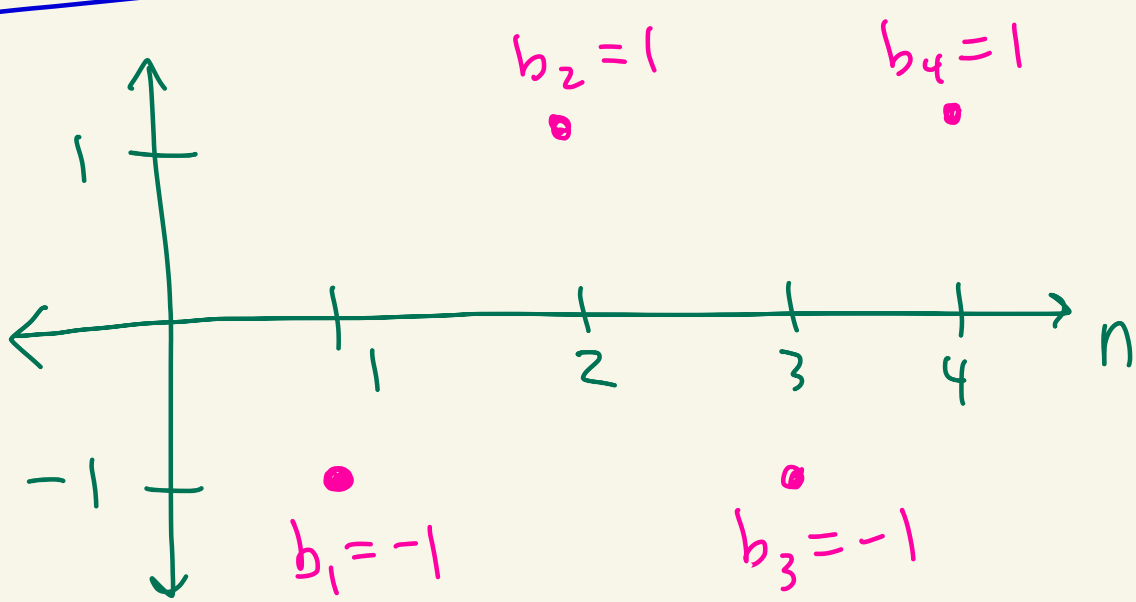
Ex: $a_n = \frac{1}{n}$, $n \geq 1$

sequence: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$



Ex:

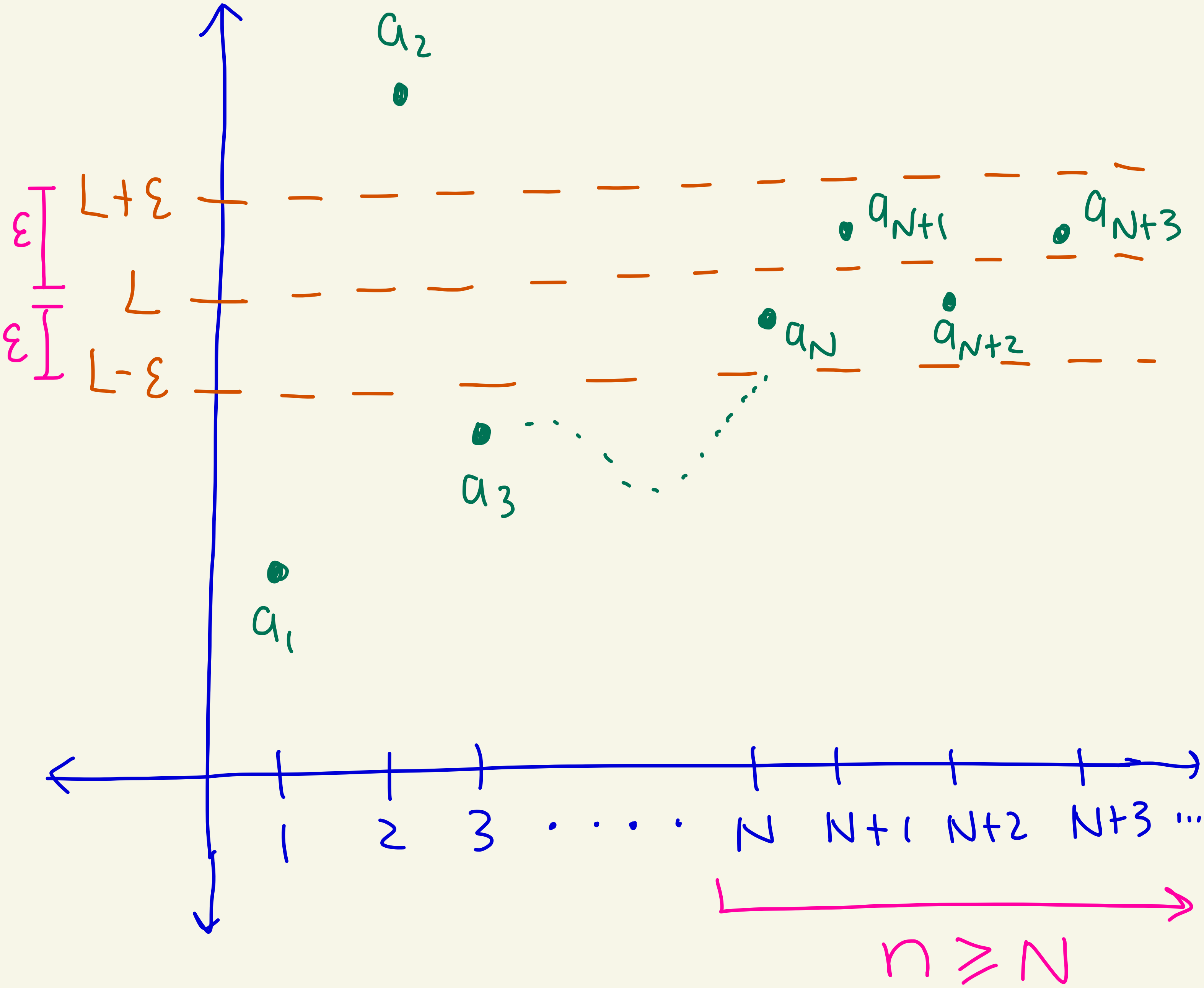
$b_n = (-1)^n$



Def: A sequence of real numbers (a_n) is said to **converge** to a limit $L \in \mathbb{R}$ if for every $\varepsilon > 0$, there exists a natural number N where if $n \geq N$, then $|a_n - L| < \varepsilon$.

If this is the case then we write $\lim_{n \to \infty} a_n = L$

If no such L exists then we say that (a_n) **diverges**.



Note: N depends on ε. You get a different N for each ε. Some people write $N(\varepsilon)$ instead of N, but we won't.