Math 4650

9/29/25

Schedule

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Ex: The harmonic series is

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

Note $\lim_{n \to \infty} \frac{1}{n} = 0$, however we will show that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Let's prove $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Let

$$S_k = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}$$

be the k-th partial sum.

We will show (S_k) does not converge.

We will do this by showing that (s_k) is not a Cauchy sequence.

If $m > n$, then

$$|s_m - s_n|$$

$$= \left| \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{m}\right) - \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right) \right|$$

$$= \left| \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{m} \right|$$

$$\left| \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) - \left(1 + \frac{1}{2}\right) \right| = \left| \frac{1}{3} + \frac{1}{4} \right|$$

$m = 4, n = 2$

$$= \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{m}$$

$$> \frac{1}{m} + \frac{1}{m} + \dots + \frac{1}{m}$$

$$= \frac{m-n}{m} = 1 - \frac{n}{m}$$

So, if $m > n$,

then $|s_m - s_n| > 1 - \frac{n}{m}$

In particular, if $m = 2n$, then

$$|s_m - s_n| > 1 - \frac{n}{2n} = 1 - \frac{1}{2} = \frac{1}{2}.$$

Why does this make the sequence not Cauchy?

If (s_k) was Cauchy then there would exist $N > 0$ where if $m, n \geq N$, then

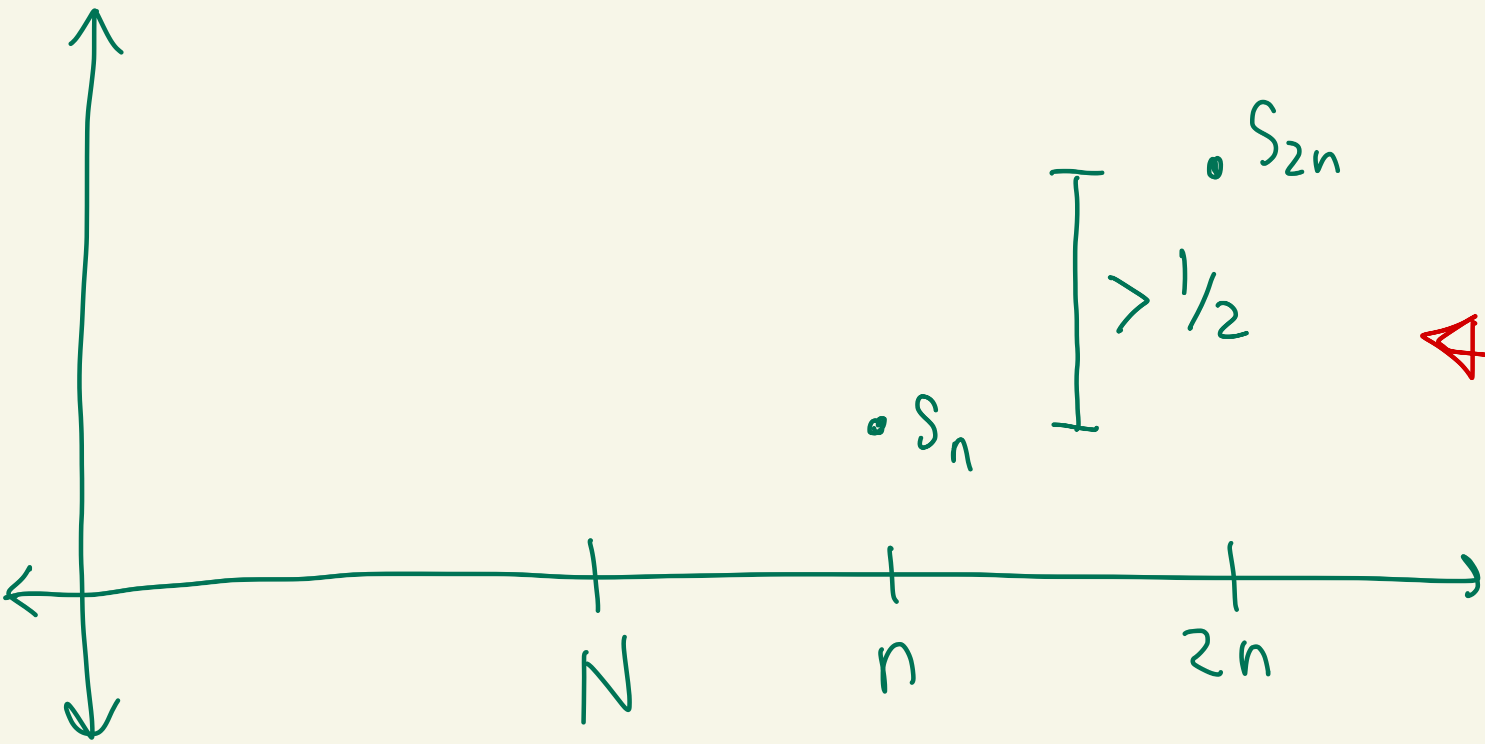
$$|s_m - s_n| < \underbrace{\frac{1}{2}}_{\varepsilon}$$

But from above if we pick

$n \geq N$ and $m = 2n \geq N$,

then $|s_m - s_n| > \frac{1}{2}$.

Contradiction.



Thus, (s_k) diverges and so does $\sum_{n=1}^{\infty} \frac{1}{n}$.

$\blacksquare$

Next we look at $\sum_{n=1}^{\infty} \frac{1}{n^p}$

where $p > 1$.

<u>Lemma:</u> Suppose (a_n) is monotonically increasing.
If there exists a subsequence (a_{n_k}) that is bounded from above, then (a_n) is bounded.

<u>proof:</u>
We know that

$$a_1 \le a_2 \le a_3 \le a_4 \le a_5 \le \cdots$$

A lower bound for (a_n) is a_1.
Let's get an upper bound.

We are given that there exists a subsequence (a_{n_k}) thats bounded from above.

So we have

$$a_{n_1} \leq a_{n_2} \leq a_{n_3} \leq \cdots$$

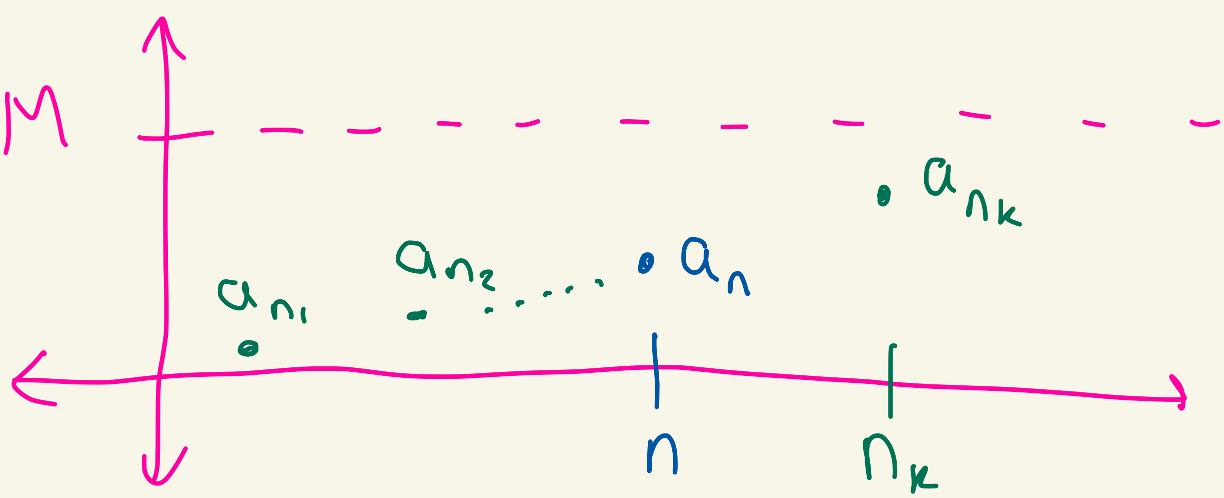
Where $n_1 < n_2 < n_3 < \cdots$

where $a_{n_k} < M$ for all n_k.

Let $n \geq 1$.

Pick some $n_k > n$.

Then, $a_n \leq a_{n_k} < M$.



So, $a_1 \leq a_n \leq M$ for all $n \geq 1$. $\blacksquare$

$$\boxed{\text{\underline{Theorem:} If } p > 1, \text{ then } \sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges.}}$$

<u>proof:</u>

Let $S_k = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \dots + \frac{1}{k^p}$

be the k-th partial sum.

So,

$S_1 = 1$

$S_2 = 1 + \frac{1}{2^p}$

$$S_3 = 1 + \frac{1}{2^p} + \frac{1}{3^p}$$

and so on...

We have

$$S_1 < S_2 < S_3 < S_4 < \cdots$$

So, (S_k) is monotonically increasing.

Let's show that (S_k) is a bounded sequence.

Then by the monotone convergence theorem it must converge.

We use the lemma to bound (S_k)

We find a subsequence thats bounded from above.

Let $k_1 = 2^1 - 1 = 1$

Set $S_{k_1} = S_1 = \frac{1}{1^p} = 1.$

Let $k_2 = 2^2 - 1 = 3.$

Set $S_{k_2} = S_3 = \frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{3^p}\right)$

$$< \frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{2^p}\right)$$

$\boxed{\begin{array}{c} 2^p < 3^p \\ \text{since } p > 1 \end{array}}$

$$= 1 + \frac{2}{2^p} = 1 + \frac{1}{2^{p-1}}$$

Let $k_3 = 2^3 - 1 = 7.$

Set $S_{k_3} = \frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right)$

$$< 1 + \left(\frac{1}{2^p} + \frac{1}{2^p}\right) + \left(\frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}\right)$$

$$= 1 + \frac{2}{2^p} + \frac{4}{4^p}$$

$$= 1 + \frac{1}{2^{p-1}} + \left(\frac{1}{2^{p-1}}\right)^2$$

In general, set $k_j = 2^j - 1$ and $r = \frac{1}{2^{p-1}}$. We get:

$$S_{k_j} < 1 + \frac{1}{2^{p-1}} + \left(\frac{1}{2^{p-1}}\right)^2 + \cdots + \left(\frac{1}{2^{p-1}}\right)^{j-1}$$

$$= 1 + r + r^2 + \cdots + r^{j-1}$$

$$= \frac{1 - r^j}{1 - r}$$

$$< \frac{1}{1-r} = \frac{1}{1 - 1/2^{p-1}}$$

Thus, (S_{k_j}) is a bounded

subsequence. By lemma, and monotone convergence thm (s_k) will converge. $\boxed{//}$

Ex: It can be shown that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{|B_{2k}|(2\pi)^{2k}}{2(2k)!},$$

B_{2k} is a Bernoulli number

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \approx 1.202057\ldots \leftarrow \boxed{\text{Apery's constant}}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^5} \approx 1.0369278\ldots$$