Math 4650

9/24/25

Topic 3 - Infinite Series

Def: Suppose we have a sequence $(a_n)_{n=1}^{\infty}$ and we want to make an infinite series

$$a_1 + a_2 + a_3 + a_4 + \cdots$$

out of it.
Define the partial sums by

$$s_k = a_1 + a_2 + \cdots + a_k$$

where $k \geq 1$.
So,

$$s_1 = a_1$$
$$s_2 = a_1 + a_2$$
$$s_3 = a_1 + a_2 + a_3$$
$$\vdots \qquad \vdots$$

If $\lim_{k\to\infty} S_k$ exists and equals L

then we say that the series

$\sum_{n=1}^{\infty} a_n$ <u>converges</u> and $\sum_{n=1}^{\infty} a_n = L$.

If $\lim_{k\to\infty} S_k$ does not exist, then

we say that $\sum_{n=1}^{\infty} a_n$ <u>diverges</u>.

<u>Note:</u> The series can start somewhere not at $n=1$, such as

$$\sum_{n=3}^{\infty} a_n = a_3 + a_4 + a_5 + a_6 + \cdots$$

Then define

$$S_3 = a_3, \quad S_4 = a_3 + a_4, \quad S_5 = a_3 + a_4 + a_5$$

and so on.

Ex: Consider

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$$

Let's calculate some partial sums.

k	$S_k = 1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^k}$
0	$S_0 = 1$
1	$S_1 = 1 + \frac{1}{2} = 1.5$
2	$S_2 = 1 + \frac{1}{2} + \frac{1}{4} = 1.75$
3	$S_3 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 1.875$
4	$S_4 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = 1.9375$
$\vdots$	$\vdots$
50	$S_{50} = 1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^{50}}$

	$\approx 1.\underbrace{999\ldots99}_{15\ 9\text{'s}}111182\ldots$
$\vdots$	$\vdots$
100	$S_{100} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{100}}$ $\approx 1.\underbrace{999\ldots99}_{30\ 9\text{'s}}11139\ldots$
$\vdots$	$\vdots$

It looks like $\lim\limits_{k\to\infty} s_k = 2$

If so, then $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n} = 2$

We will show this

for $\displaystyle\sum_{n=0}^{\infty} r^n$ when $-1 < r < 1$.

We need:

$\boxed{\text{HW 3}}$ If $r \in \mathbb{R}$ and $|r|<1$, then $\lim_{n\to\infty} r^n = 0$

Ex: $\lim_{n\to\infty} \left(\frac{1}{2}\right)^n = 0$ $\leftarrow$ $\boxed{r = 1/2}$

Ex: (Geometric series)

We are interested in

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + r^4 + \cdots$$

where $|r|<1$. $\leftarrow$ $\boxed{-1<r<1}$

Let $S_k = 1 + r + r^2 + \cdots + r^k$

be the k-th partial sum.

Note that

$$s_k(1-r) = s_k - r s_k$$
$$= 1 + r + r^2 + \cdots + r^k$$
$$- r - r^2 - \cdots - r^k - r^{k+1}$$
$$= 1 - r^{k+1}$$

So,

$$s_k = \frac{1-r^{k+1}}{1-r}$$

Thus,

$$\lim_{k\to\infty} s_k = \lim_{k\to\infty} \frac{1-r^{k+1}}{1-r}$$

$$= \frac{\lim_{k\to\infty}(1-r^{k+1})}{\lim_{k\to\infty}(1-r)}$$

$$= \frac{\lim_{k\to\infty} 1 - \lim_{k\to\infty} r^{k+1}}{1-r}$$

$$= \frac{1-0}{1-r} = \frac{1}{1-r}$$

Thus, if $|r|<1$, then

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

<u>Ex:</u> $\displaystyle \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2$

$$\boxed{\begin{array}{c} r = \frac{1}{2} \\ -1 < r < 1 \end{array}}$$

Ex: Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \underbrace{\frac{1}{1\cdot 2}}_{n=1} + \underbrace{\frac{1}{2\cdot 3}}_{n=2} + \underbrace{\frac{1}{3\cdot 4}}_{n=3} + \cdots$$

The trick is to write

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

multiply by $n(n+1)$ to get:

$$1 = A(n+1) + Bn$$

This has to work for all n.

$\underline{n=-1:}$ $1 = A(0) - B$

$-1 = B$

$\underline{n=0:}$ $1 = A(0+1) + B(0)$

$1 = A$

Thus, $\dfrac{1}{n(n+1)} = \dfrac{1}{n} + \dfrac{-1}{n+1}$

$\boxed{\begin{array}{l}\underline{\text{Another way:}} \\ 1 = A(n+1) + Bn \\ 1 = \underbrace{(A+B)}_{0}n + \underbrace{A}_{1} \\ \left.\begin{array}{l} A = 1 \\ A+B = 0 \end{array}\right] \to \begin{array}{l} A = 1 \\ B = -1 \end{array}\end{array}}$

So,

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$$

Let's look at the partial sums.

$$s_1 = \underbrace{\left(\frac{1}{1} - \frac{1}{2} \right)}_{n=1} = 1 - \frac{1}{2}$$

$$S_2 = \underbrace{\left(1 - \cancel{\tfrac{1}{2}}\right)}_{n=1} + \underbrace{\left(\cancel{\tfrac{1}{2}} - \tfrac{1}{3}\right)}_{n=2} = 1 - \frac{1}{3}$$

$$S_3 = \underbrace{\left(1 - \cancel{\tfrac{1}{2}}\right)}_{n=1} + \underbrace{\left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right)}_{n=2} + \underbrace{\left(\cancel{\tfrac{1}{3}} - \tfrac{1}{4}\right)}_{n=3} = 1 - \frac{1}{4}$$

In general,

$$S_k = 1 - \frac{1}{k+1}$$

Thus,

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{k\to\infty} S_k = \lim_{k\to\infty}\left(1 - \frac{1}{k+1}\right)$$

$$= 1 - 0 = 1$$

So,

$$1 = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots$$

Theorem (Divergence Theorem)

If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$

$\boxed{\text{Contrapositive: If } \lim_{n\to\infty} a_n \neq 0, \text{ then } \sum a_n \text{ diverges}}$

Prove:

Suppose $\sum_{n=1}^{\infty} a_n$ converges to L.

Then $\lim_{k\to\infty} S_k = L$ where S_k is the k-th partial sum.

Note that

$$a_n = (a_1 + a_2 + \dots + a_n) - (a_1 + a_2 + \dots + a_{n-1})$$

$$= S_n - S_{n-1}$$

Thus,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(S_n - S_{n-1}\right)$$

$$= \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$$

$$= L - L$$

$$= 0 \qquad \blacksquare$$

Ex: Consider

$$\sum_{n=1}^{\infty} \frac{n}{n+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \cdots$$

Note

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1 + 1/n} = \frac{1}{1+0} = 1$$

divide by n top/bottom

Since $\lim_{n\to\infty} \frac{n}{n+1} \neq 0$,

the series $\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges