Math 4650
9/22/25

Topic 2a - Application of Monotone Convergence Theorem

Let $a \in \mathbb{R}$ with $a > 0$.
You can find a monotonically decreasing sequence (a_n) that converges to $\sqrt{a}$.

Let $a_1 > 0$ be any real number.
Let $\underbrace{a_{n+1} = \frac{1}{2}\left(a_n + \frac{a}{a_n}\right), \ n \geq 1.}_{\text{this is from Newton's method.}}$

Let's approximate $\sqrt{2}$.

Here $a = 2$.

Let $a_1 = 1 > 0$.

Then, $a_2 = \frac{1}{2}\left(a_1 + \frac{2}{a_1}\right)$

$= \frac{1}{2}\left(1 + \frac{2}{1}\right) = \frac{3}{2} = 1.5$

And, $a_3 = \frac{1}{2}\left(a_2 + \frac{2}{a_2}\right)$

$= \frac{1}{2}\left(\frac{3}{2} + \frac{2}{(3/2)}\right) = \frac{17}{12} \approx 1.41\overline{6}$

And, $a_4 = \frac{1}{2}\left(a_3 + \frac{2}{a_3}\right)$

$= \frac{1}{2}\left(\frac{17}{12} + \frac{2}{(17/12)}\right) = \frac{577}{408}$

$\approx 1.41421568\ldots$

And, $a_5 = \frac{1}{2}\left(a_4 + \frac{2}{a_4}\right)$

$$= \frac{1}{2}\left(\frac{577}{408} + \frac{2}{\left(\frac{577}{408}\right)}\right)$$

$$= \frac{665857}{470832} \approx \underline{1.4142135623}7\ldots$$

Note:

$$\sqrt{2} \approx \underline{1.41421356237}309\ldots$$

Theorem: Let $a > 0$.

Define: $a_1 > 0$ is any real number

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{a}{a_n}\right) \text{ for } n \geq 1$$

Then:

1. $(a_n)_{n=1}^{\infty}$ converges
2. $\lim\limits_{n \to \infty} a_n = \sqrt{a}$
3. $|a_n - \sqrt{a}| \leq \dfrac{a_n^2 - a}{a_n}$ when $n \geq 2$

$\boxed{\text{error bound}}$

Proof:

① Break this into 3 facts.

Fact 1: $a_n > 0$ for $n \geq 1$

We know $a_1 > 0$.

If $a_k > 0$, then

$$a_{k+1} = \frac{1}{2}\left(a_k + \frac{a}{a_k}\right) > 0$$

By induction, $a_n > 0$ for $n \geq 1$

Fact 2: $a_n \geq \sqrt{a}$ for $n \geq 2$

Let $k \geq 1$.

By def $2a_{k+1} = a_k + \frac{a}{a_k}$

So, $a_k^2 - 2a_k a_{k+1} + a = 0$

Thus, $x^2 - 2a_{k+1}x + a = 0$

has a real root $x = a_k$

Thus, the discriminant is not negative.

So,

$$(-2a_{k+1})^2 - 4(1)(a) \geq 0$$

Thus,

$$4a_{k+1}^2 - 4a \geq 0$$

So, $a_{k+1}^2 \geq a$

Thus,

$$a_{k+1} \geq \sqrt{a}$$

for $k \geq 1$

Hence,

$$a_n \geq \sqrt{a} \quad \text{for } n \geq 2$$

Fact 3: $a_n \geq a_{n+1}$ for $n \geq 2$

Let $n \geq 2$.

Then,

$$a_n - a_{n+1} = a_n - \frac{1}{2}\left(a_n + \frac{a}{a_n}\right)$$

$$= \frac{1}{2} a_n - \frac{1}{2}\frac{a}{a_n}$$

$$= \frac{1}{2}\left(\frac{a_n^2 - a}{a_n}\right) \geq 0$$

Fact 2:
$a_n \geq \sqrt{a}$
$a_n^2 - a \geq 0$
Fact 1:
$a_n > 0$

So, $a_n - a_{n+1} \geq 0$

Thus, $a_n \geq a_{n+1}$.

Now we finish part 1

We have from above that:

$$a_2 \geq a_3 \geq a_4 \geq a_5 \geq \cdots\cdots \geq \sqrt{a} > 0$$

So we have a sequence thats bounded between a_2 and $\sqrt{a}$ and its monotonically decreasing.

So, by the monotone convergence theorem, $(a_n)_{n=1}^{\infty}$ converges

$\boxed{1}$

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2 Let's show $\lim_{n \to \infty} a_n = \sqrt{a}$

We know $L = \lim_{n \to \infty} a_n$ exists.

We know $a_{n+1} = \frac{1}{2}\left(a_n + \frac{a}{a_n}\right)$

Thus, $\lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} \frac{1}{2}\left(a_n + \frac{a}{a_n}\right)$

So, $L = \frac{1}{2}\left(L + \frac{a}{L}\right)$

Thus, $L^2 = \frac{1}{2}L^2 + \frac{1}{2}a$

So, $2L^2 - L^2 = a$

Thus, $L^2 = a$

So, $L = \pm\sqrt{a}$

We know $L \geq 0$ because $a_n \geq 0$ for all n. HW 2

So, $L = \sqrt{a}$

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3 Online

HW 2

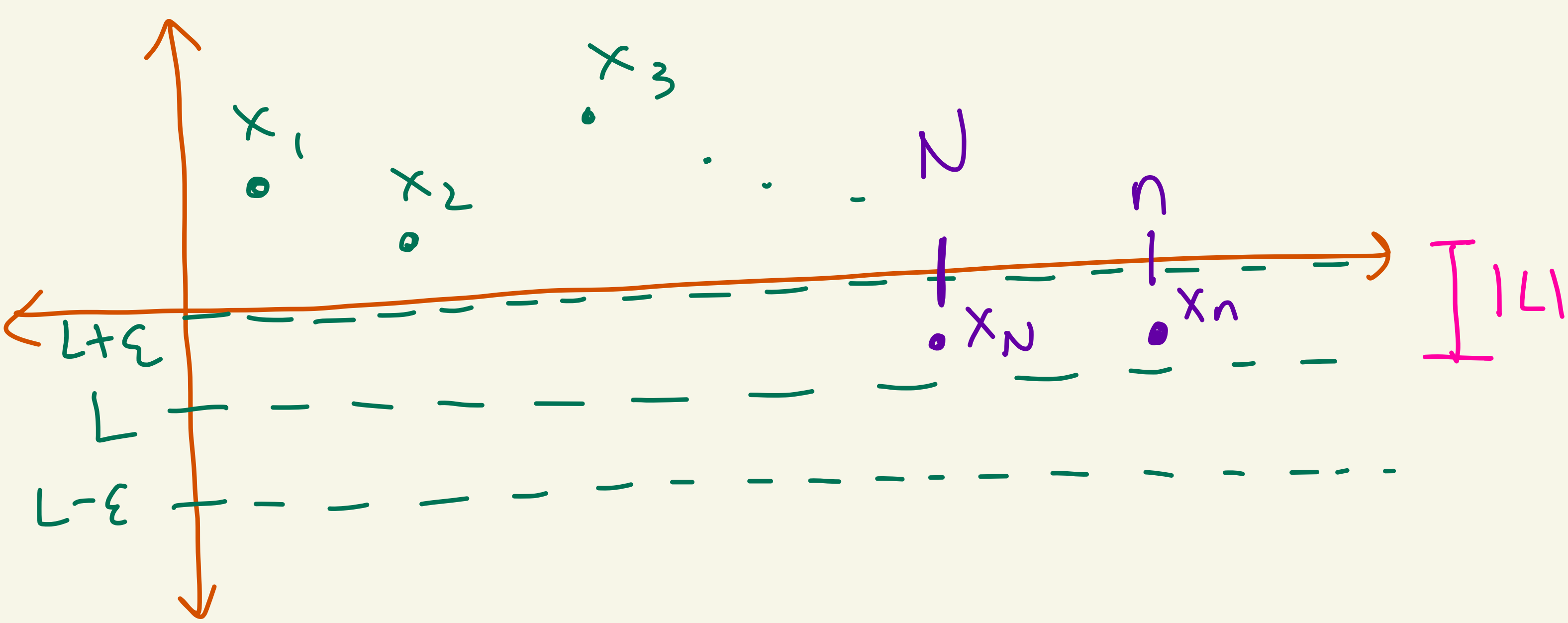
6 (a) Suppose $(x_n)_{n=1}^{\infty}$ is a convergent sequence with $x_n \geq 0$ for $n \geq 1$.

If $\lim_{n \to \infty} x_n = L$, then $L \geq 0$.

<u>Proof:</u>

Let's prove this by contradiction.

Suppose $L < 0$.



Let $\varepsilon = |L| > 0$

$L \neq 0$

Since $\lim_{n \to \infty} x_n = L$ there exists

$N > 0$ where if $n \geq N$

then $|x_n - L| < \varepsilon$

In particular $|x_N - L| < \varepsilon$.

So, $|x_N - L| < \underbrace{|L|}_{\varepsilon}$

Then,

$$-|L| < x_N - L < |L|$$

$\boxed{\begin{array}{c} |y| < c \\ \text{iff} \\ -c < y < c \end{array}}$

So,

$$L - |L| < x_N < L + |L|$$

Since $L<0$ we know $|L| = -L$.

Thus,

$$L-(-L) < x_N < L-L$$

So,

$$2L < x_N < 0$$

Thus, $x_N < 0$.

But the assumption was $x_n \geq 0$ for all n.

Contradiction.

Thus, $L<0$ can't happen.

Hence, $L \geq 0$. $\blacksquare$