Math 4650

9/17/25

Bolzano - Weierstrass Theorem

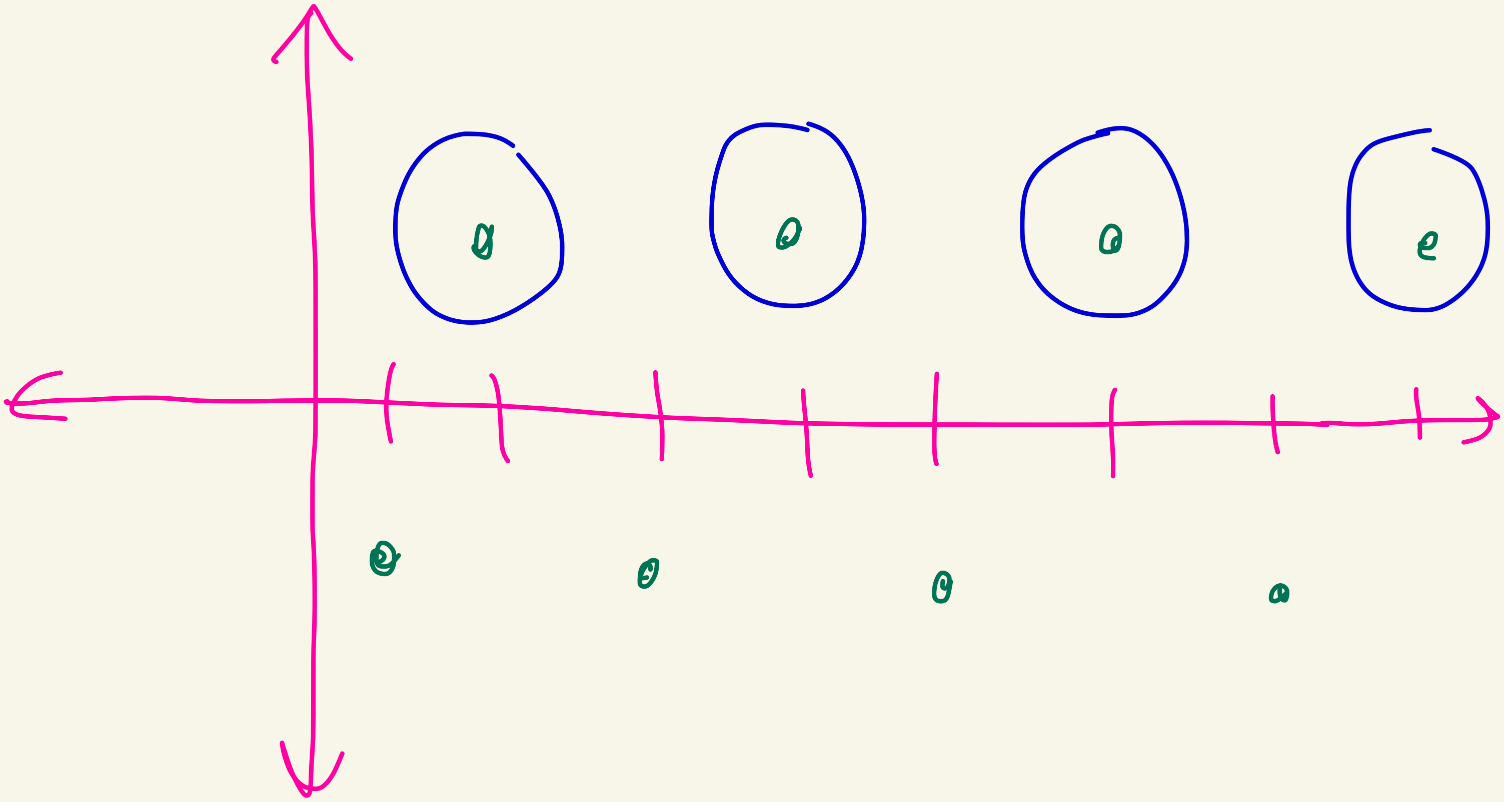
Let (a_n) be a bounded sequence. Then there exists a convergent subsequence.

Proof:

Let (a_n) be a bounded sequence. By the monotone subsequence theorem there exists a monotone subsequence (a_{n_k}).

Since (a_{n_k}) is bounded and monotone, by the monotone convergence theorem it converges. $\blacksquare$

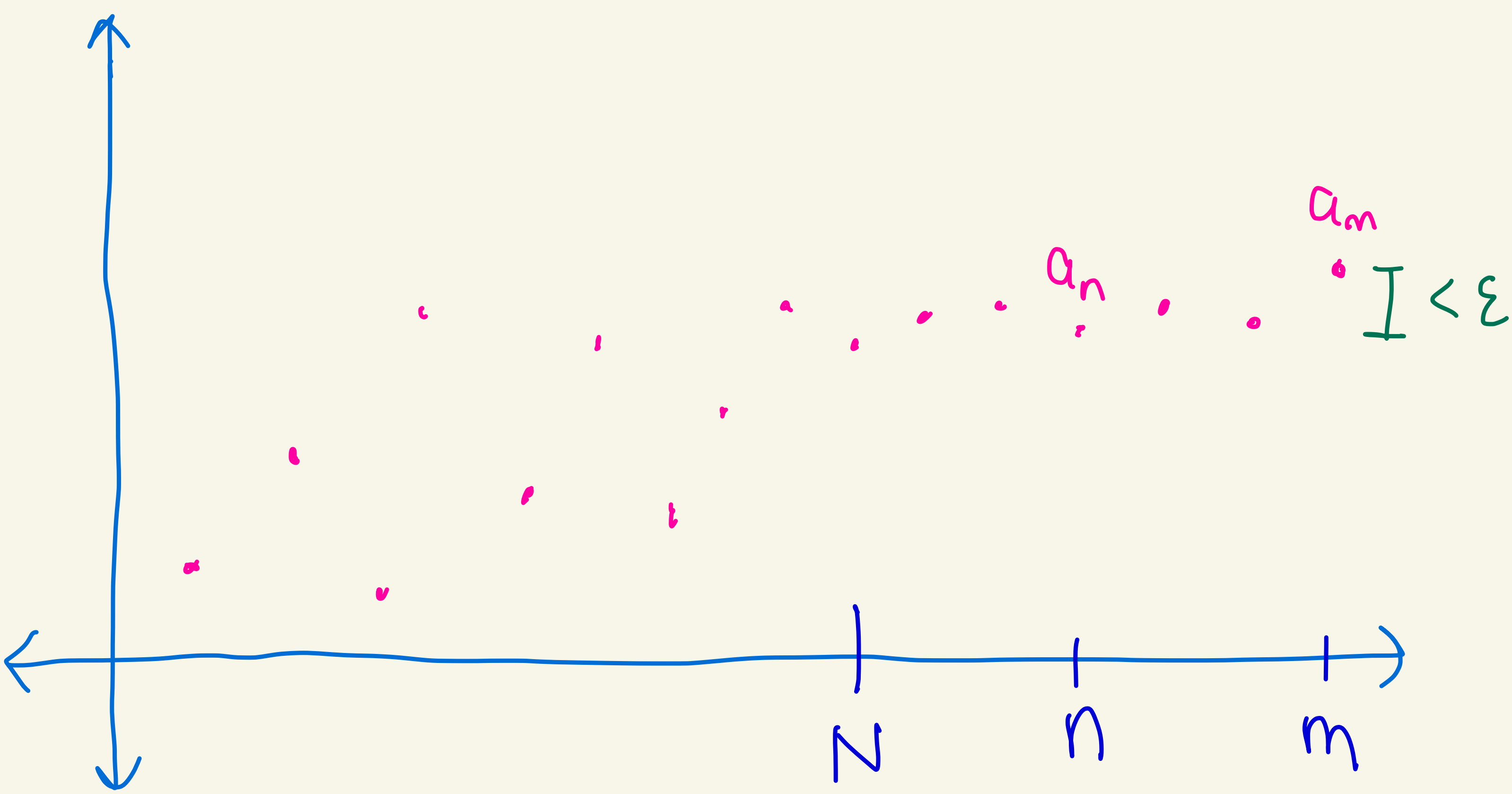
Ex: $a_n = (-1)^n$



Sequence: $-1, 1, -1, 1, -1, 1, -1, 1, \ldots$

Convergent Subsequence: $1, 1, 1, 1, 1, 1, \ldots$

Def: Let (a_n) be a sequence of real numbers. We say that (a_n) is a Cauchy sequence if for every $\varepsilon > 0$ there exists $N > 0$ where $n, m \geq N$, then $|a_n - a_m| < \varepsilon$.



Ex: Let's prove that

$$\left(\frac{1}{n}\right)_{n=1}^{\infty}$$ is Cauchy.

proof:

$|x+y| \le |x|+|y|$
Δ-inequality

Let $\varepsilon > 0$.

Note that

$$\left|\frac{1}{n} - \frac{1}{m}\right| \le \left|\frac{1}{n}\right| + \left|-\frac{1}{m}\right|$$

$$= \frac{1}{n} + \frac{1}{m}$$

Pick $N > \frac{2}{\varepsilon}$.

Then if $n, m \geq N$, then

$$\left|\frac{1}{n} - \frac{1}{m}\right| \le \frac{1}{n} + \frac{1}{m} \le \frac{1}{N} + \frac{1}{N}$$

$$\boxed{n, m \geq N}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$= \varepsilon$$

So, if $n, m \geq N$, then

$$\left| \frac{1}{n} - \frac{1}{m} \right| < \varepsilon.$$

So, the sequence is a Cauchy sequence. ∎

Theorem: Let (a_n) be a sequence of real numbers. Then, (a_n) converges if and only if (a_n) is a Cauchy sequence.

proof:

($\Rightarrow$) Suppose (a_n) converges.

So, $\lim\limits_{n\to\infty} a_n = L$ for some $L \in \mathbb{R}$

Let $\varepsilon > 0$.

Then there exists $N > 0$ where if $k \geq N$ then

$\boxed{|a_k - L| < \varepsilon/2}$.

Then if $n, m \geq N$, we have

$$|a_n - a_m| = |a_n - L + L - a_m|$$

$$\leq |a_n - L| + |L - a_m|$$

Δ-inequality

$$= |a_n - L| + |a_m - L|$$

$$< \varepsilon/2 + \varepsilon/2$$

$= \varepsilon$

So if $n, m \geq N$, then $|a_n - a_m| < \varepsilon$

Thus, (a_n) is a Cauchy sequence.

($\Leftarrow$) Suppose (a_n) is a Cauchy sequence.

Let's show it must converge.

By HW 2 #9, since (a_n) is a Cauchy sequence it must be bounded.

By Bolzano-Weierstrass, there exists a convergent subsequence (a_{n_k}) where $\lim_{k \to \infty} a_{n_k} = L$

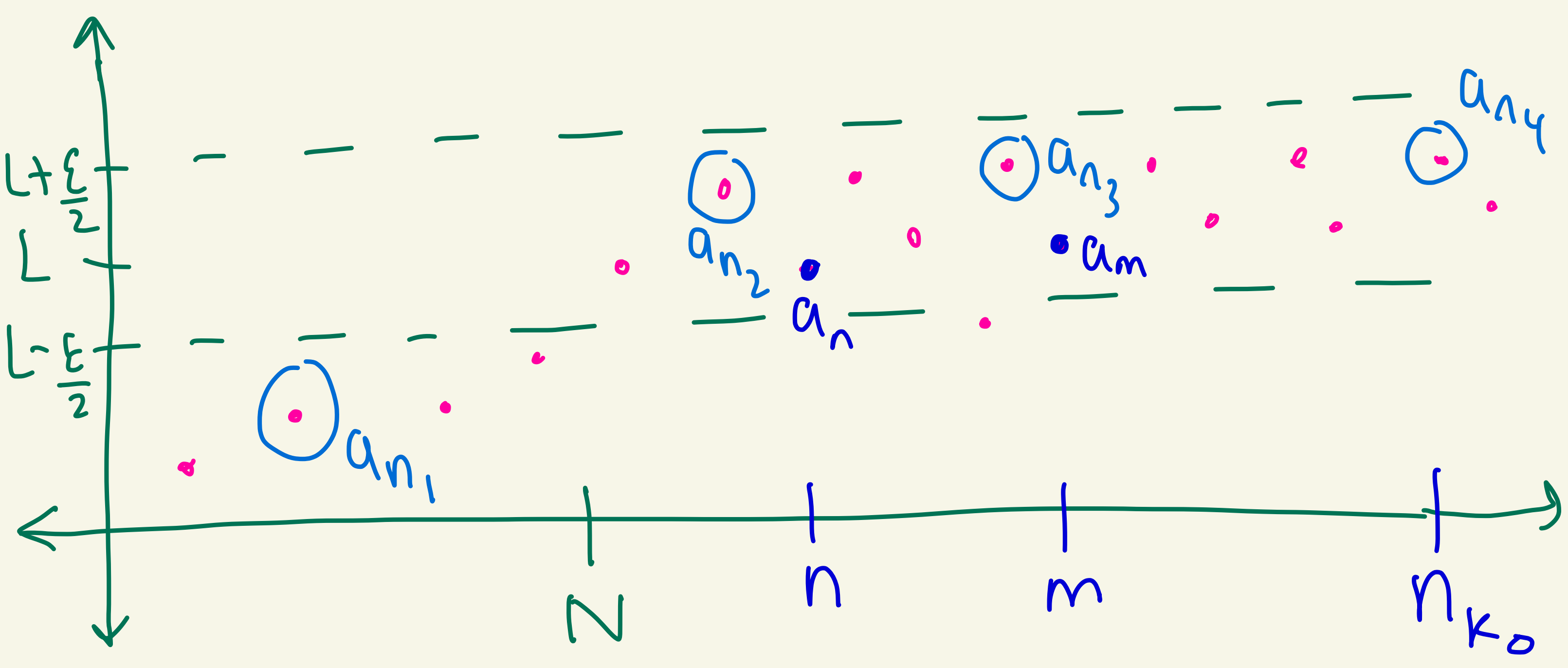
for some $L \in \mathbb{R}$.

We will show that $\lim_{n\to\infty} a_n = L$.

Let $\varepsilon > 0$.

Since (a_n) is a Cauchy sequence there exists $N > 0$ where $n, m \geq N$ then $|a_n - a_m| < \frac{\varepsilon}{2}$

Since (a_{n_k}) converges to L there exists $n_{k_0} \geq N$ where $|a_{n_{k_0}} - L| < \varepsilon/2$.



So if $n \geq N$, then

$$|a_n - L| = |a_n - a_{n_{k_0}} + a_{n_{k_0}} - L|$$

$$\leq |a_n - a_{n_{k_0}}| + |a_{n_{k_0}} - L|$$

$\boxed{n, n_{k_0} \geq N}$

$$< \frac{\varepsilon}{2} + \varepsilon/2$$

$$= \varepsilon$$

Thus, if $n \geq N$, then $|a_n - L| < \varepsilon$

So, $\lim\limits_{n \to \infty} a_n = L$.

Therefore (a_n) converges. ∎

Note: We used the completeness axiom in ($\Leftarrow$) above.

$$
\begin{tikzcd}
\text{completeness axiom} \arrow[d] & \\
\text{monotone convergence theorem} \arrow[dr] & \text{monotone subsequence theorem} \arrow[d] \\
& \text{Bolzano-Weierstrass} \arrow[d] \\
& \text{If } (a_n) \text{ is Cauchy, then } (a_n) \text{ converges}
\end{tikzcd}
$$