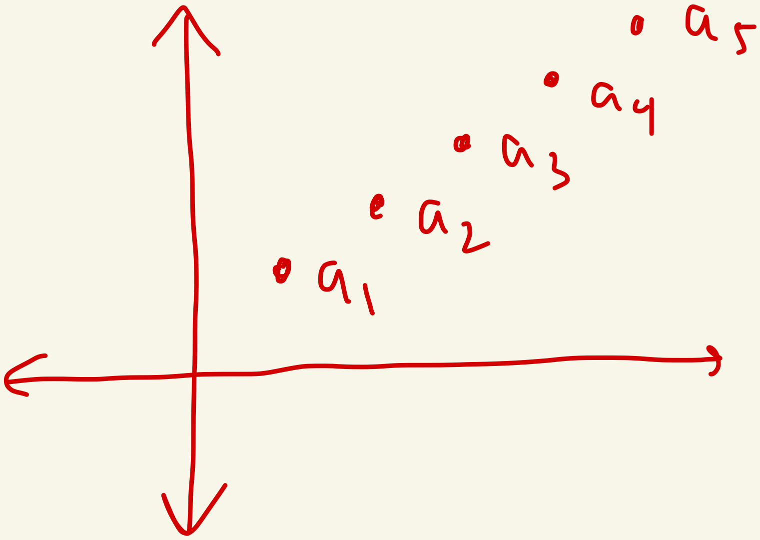
Math 4650
9/15/25

Def: Let (a_n) be a sequence of real numbers.

- We say that (a_n) is <u>monotone increasing</u> if $a_n \leq a_{n+1}$ for all n.
- We say that (a_n) is <u>monotone decreasing</u> if $a_{n+1} \leq a_n$ for all n.
- We say that (a_n) is <u>monotone</u> if it is either monotone increasing or monotone decreasing.

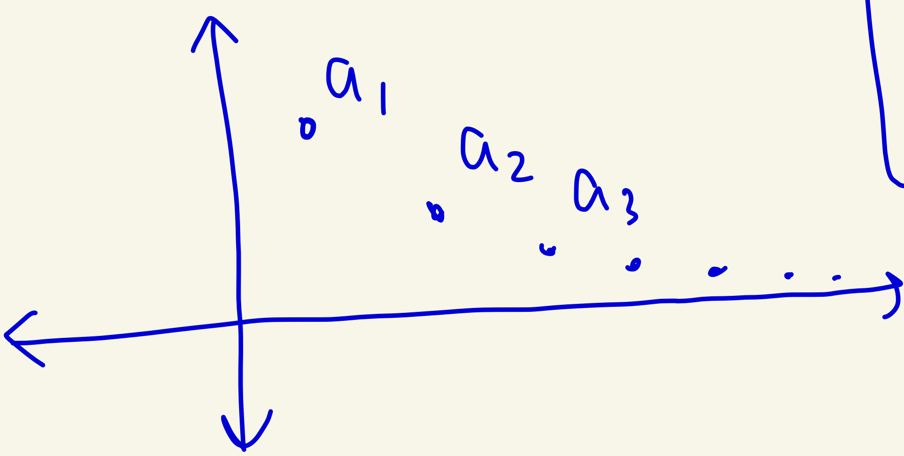
Ex: $a_n = n$



monotone increasing

monotone

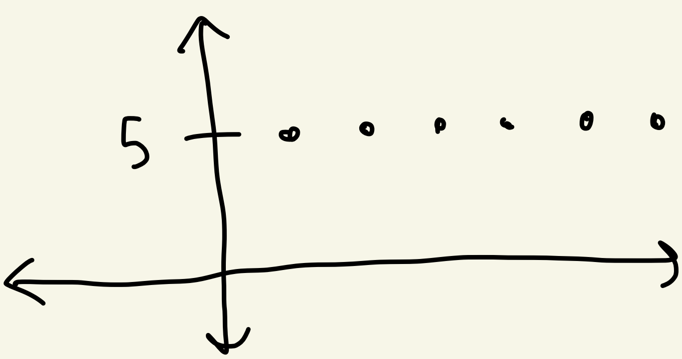
Ex: $a_n = \frac{1}{n}$



monotone decreasing

monotone

Ex: $a_n = 5$



monotone increasing

monotone decreasing

monotone

Monotone converge theorem

> If (a_n) is a bounded monotone sequence, then (a_n) converges

<u>proof</u>: We will prove this for the case when (a_n) is monotone increasing. The monotone decreasing proof is similar.

Suppose (a_n) is bounded and monotone increasing.

Since (a_n) is bounded we know there exists $M>0$ where $|a_n| \leq M$ for all n.

Since (a_n) is monotone increasing we know $a_n \leq a_{n+1}$ for all n.

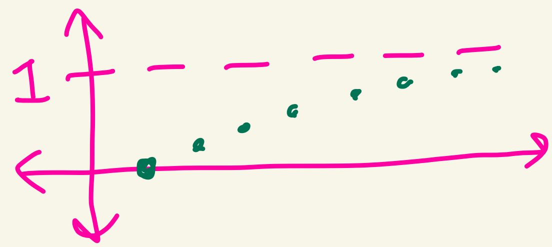
So,

$$a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5 \leq \cdots$$

Let

$$S = \{a_n \mid n \geq 1\}$$
$$= \{a_1, a_2, a_3, \ldots\}$$

Ex: $a_n = 1 - \frac{1}{n}$



$S = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\}$

$L = \sup(S) = 1$

Since S is bounded from above by M, it has a supremum

by completeness axiom

Let $L = \sup(S)$

Let's show that $\lim_{n \to \infty} a_n = L$.

Let $\varepsilon > 0$.
By the inf/sup theorem there exists $N > 0$ where

$$L - \varepsilon < \underbrace{a_N}_{\text{element of } S} \leq \underbrace{L}_{\sup(S)}$$

Since (a_n) is monotone increasing, if $n \geq N$, then $a_N \leq a_n$.

Since $L = \sup(S)$ we know that $a_n \leq L$ for all n.

Thus, if $n \geq N$, then

$$L-\varepsilon < a_N \leq a_n \leq L$$

So if $n \geq N$ then

$$L-\varepsilon < a_N \leq a_n \leq L < L+\varepsilon$$

$\varepsilon > 0$

Thus, if $n \geq N$ then

$$L-\varepsilon < a_n < L+\varepsilon$$

So, if $n \geq N$, then

$$|a_n - L| < \varepsilon.$$

Therefore $\lim_{n\to\infty} a_n = L$ ▨

Def: Let (a_n) be a sequence of real numbers. Let

$$n_1 < n_2 < n_3 < n_4 < \cdots$$

be a strictly increasing sequence of natural numbers. Then, the sequence $(a_{n_k})_{k=1}^{\infty}$ is a *subsequence* of (a_n).

Ex: $a_n = \frac{1}{n}$

sequence:

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \ldots$$

$n=1 \quad n=2 \quad n=3 \quad n=4 \quad \ldots$

subsequence:

$$\underbrace{1}_{n_1=1}, \underbrace{\frac{1}{2}}_{n_2=2}, \underbrace{\frac{1}{4}}_{n_3=4}, \underbrace{\frac{1}{8}}_{n_5=8}, \underbrace{\frac{1}{16}}_{n_6=16}, \ldots$$

Monotone subsequence theorem

If (a_n) is a sequence of real numbers, then there exists a subsequence of (a_n) that is monotone

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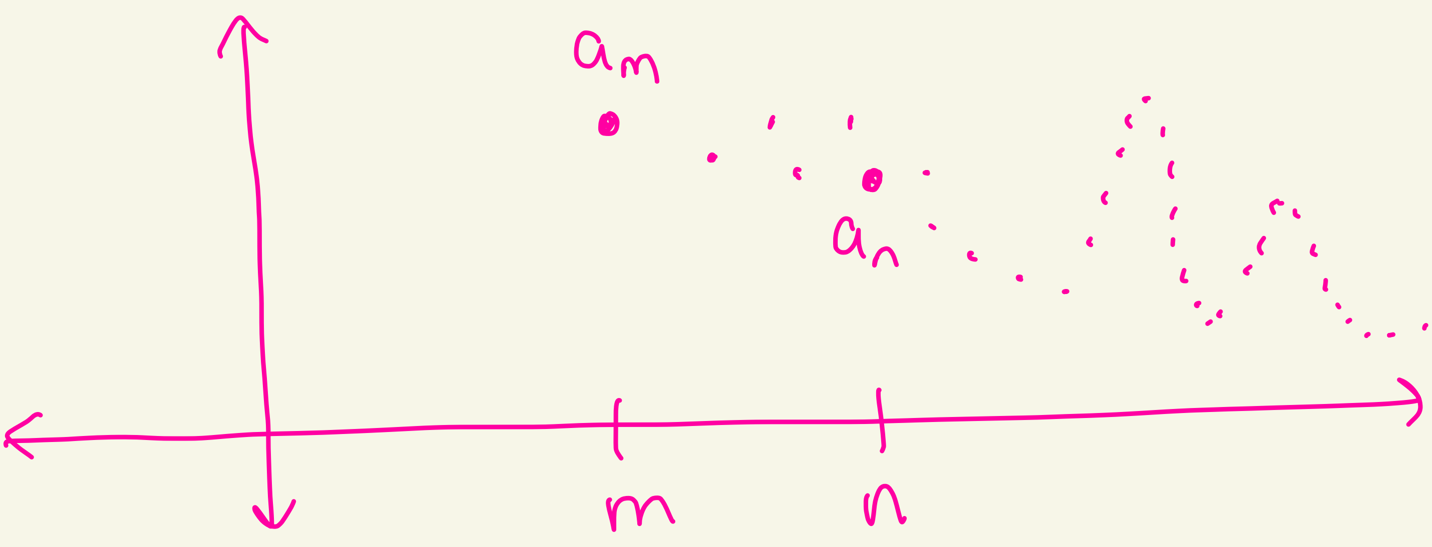
proof:

We say that the m-th term a_m is a "peak" of our sequence if $a_m \geq a_n$ for all $n \geq m$.

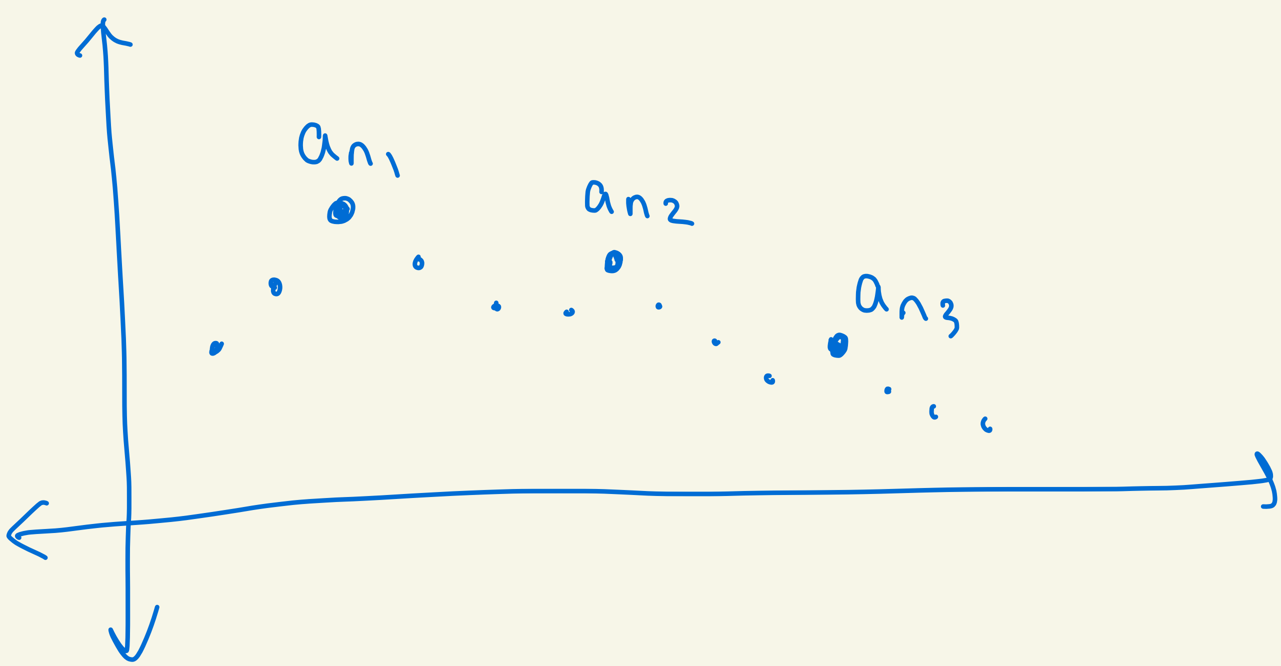


Case 1: Suppose (a_n) has infinitely many peaks. Then listing the peaks by increasing subscripts we get a subsequence of peaks:

$$a_{n_1} \geq a_{n_2} \geq a_{n_3} \geq \cdots$$

with $n_1 < n_2 < n_3 < \cdots$

So, (a_{n_k}) is a monotone decreasing subsequence.



Case 2: Suppose (a_n) has a finite number of peaks.

Set $n_1 = 1$ if there are no peaks. Otherwise set n_1 as follows:

Let the peaks be listed by increasing subscripts:

$$a_{m_1} \geqslant a_{m_2} \geqslant \cdots \geqslant a_{m_r}$$

where a_{m_r} is the last peak.

Let $n_1 = m_r + 1$.

So, a_{n_1} is the term immediately after the last peak.

So, a_{n_1} is not a peak and there are no peaks after a_{n_1}.

Thus there exists n_2 with

$n_1 < n_2$ and $a_{n_1} < a_{n_2}$.

Since a_{n_2} is also not a peak

there exists n_3 with

$n_2 < n_3$ and $a_{n_2} < a_{n_3}$.

Keep going like this to get

a subsequence

$$a_{n_1} < a_{n_2} < a_{n_3} < a_{n_4} < \cdots$$

with $n_1 < n_2 < n_3 < \ldots$

Thus, (a_{n_k}) is a monotone

increasing subsequence. $\blacksquare$