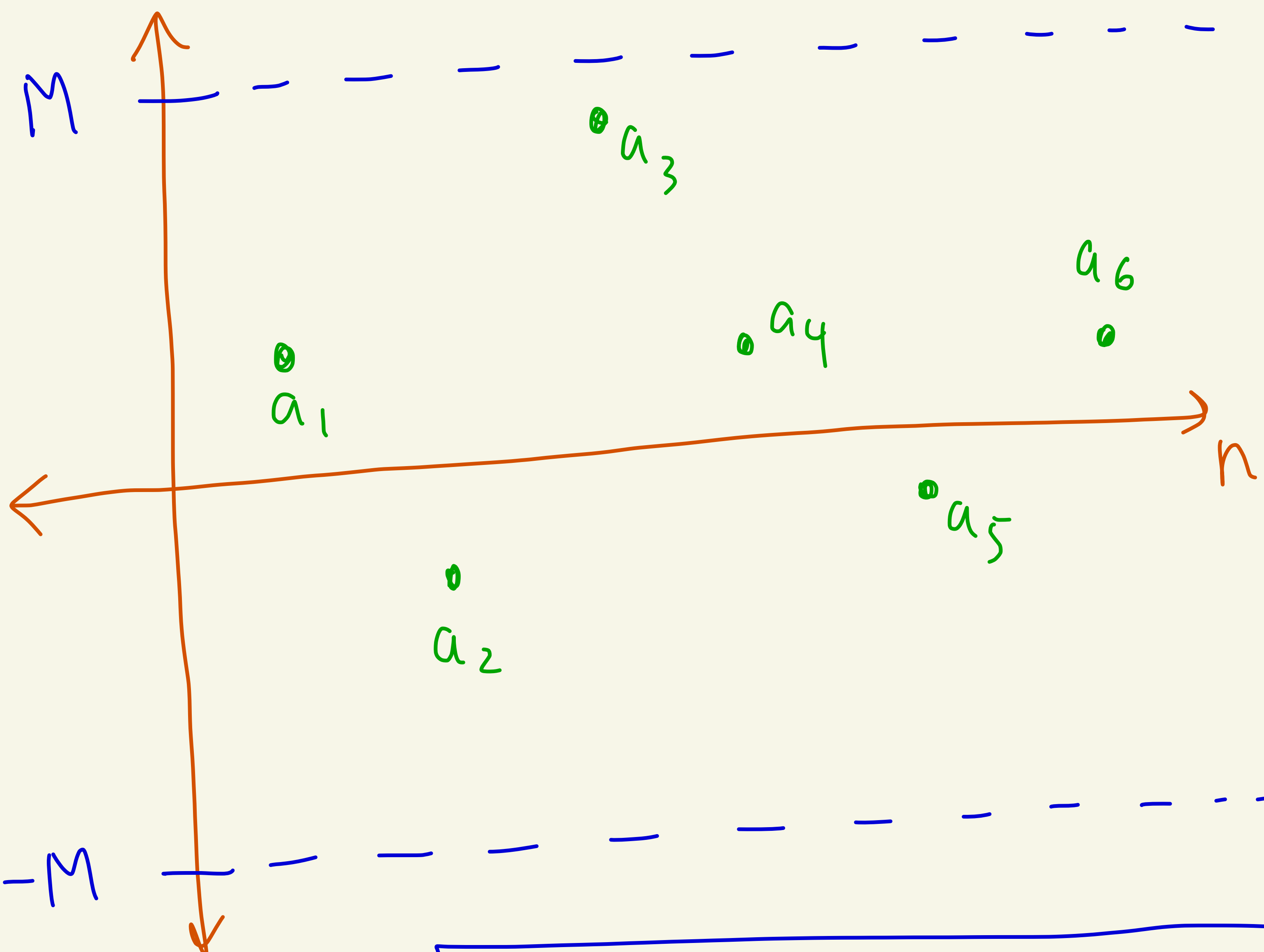
Math 4650
9/10/25

<u>Def:</u> We say that the sequence (a_n) is <u>bounded</u> if there exists $M > 0$ where $|a_n| \leq M$ for all n.



$$\boxed{|a_n| \leq M \text{ iff } -M \leq a_n \leq M}$$

Theorem: If (a_n) converges, then (a_n) is bounded.

(a_n) converges means $\lim\limits_{n\to\infty} a_n$ exists

proof:

Suppose $\lim\limits_{n\to\infty} a_n = L$.

Let $\varepsilon = 1$.

Then there exists N where if $n \geq N$, then $|a_n - L| < 1$.

So if $n \geq N$, then

$$|a_n| = |a_n - L + L|$$

$$\leq |a_n - L| + |L|$$

$$< 1 + |L|$$

Δ-inequality

$|x+y| \leq |x| + |y|$

So, if $n \geq N$,

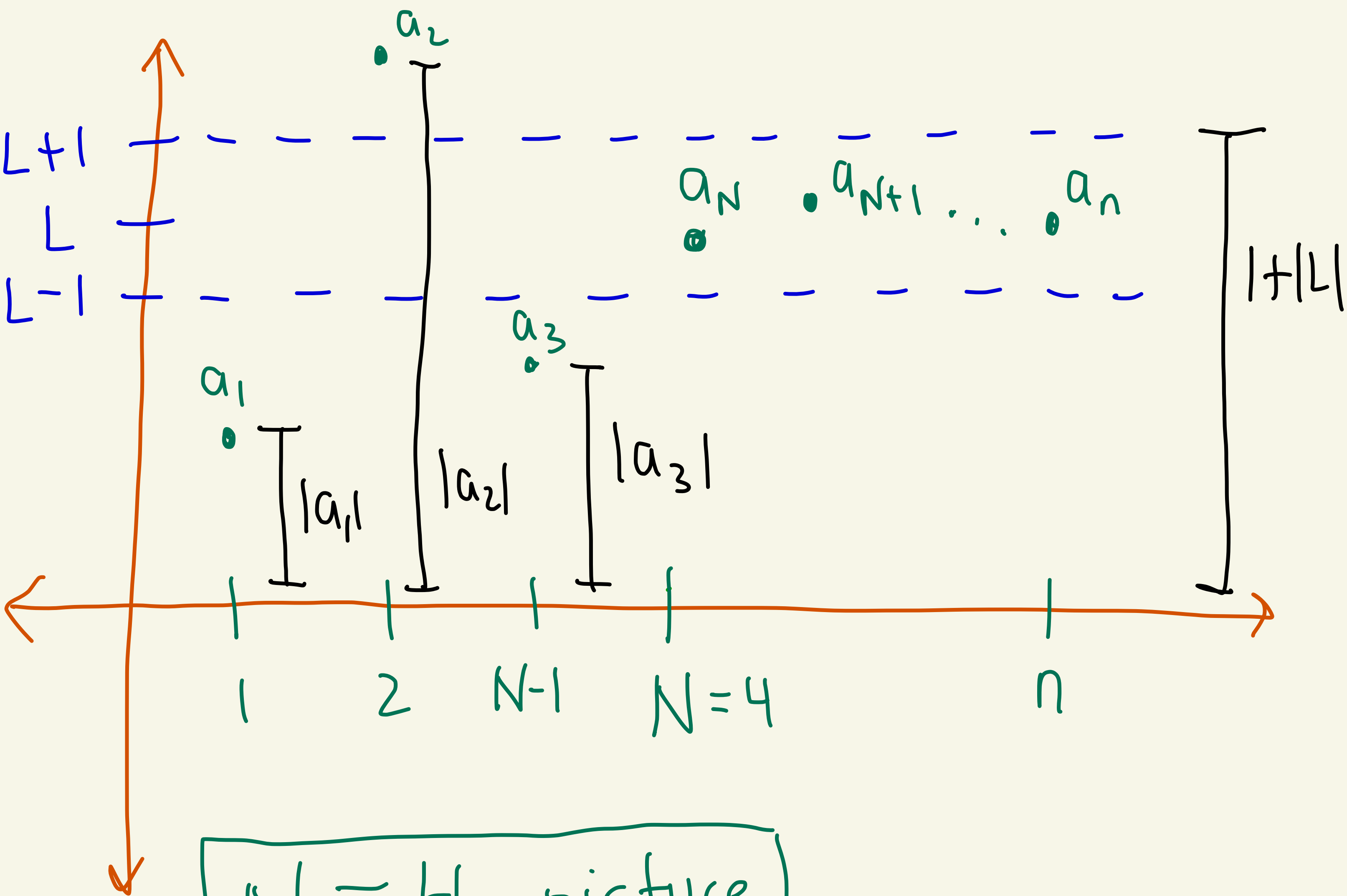
then $|a_n| < 1 + |L|$.

Let

$$M = \max\{|a_1|, |a_2|, \dots, |a_{N-1}|, 1 + |L|\}$$

Then,

$|a_n| \leq M$ for all n.



$$\boxed{N = 4 \text{ picture}}$$

In this picture

$$M = \max\{|a_1|, |a_2|, |a_3|, 1+|L|\}$$
$$= |a_2|$$

$\blacksquare$

The converse of the above is not true.

Converse: "If (a_n) is bounded, then (a_n) converges."

Ex: $(a_n) = (-1)^n$ is bounded since $|a_n| \leq 1$ but it diverges.

Theorem: If (a_n) is not bounded, then (a_n) does not converge.

pf: Contrapositive of theorem above. $\blacksquare$

Ex:

Let's show that (n^2) diverges.

We show that its not bounded.

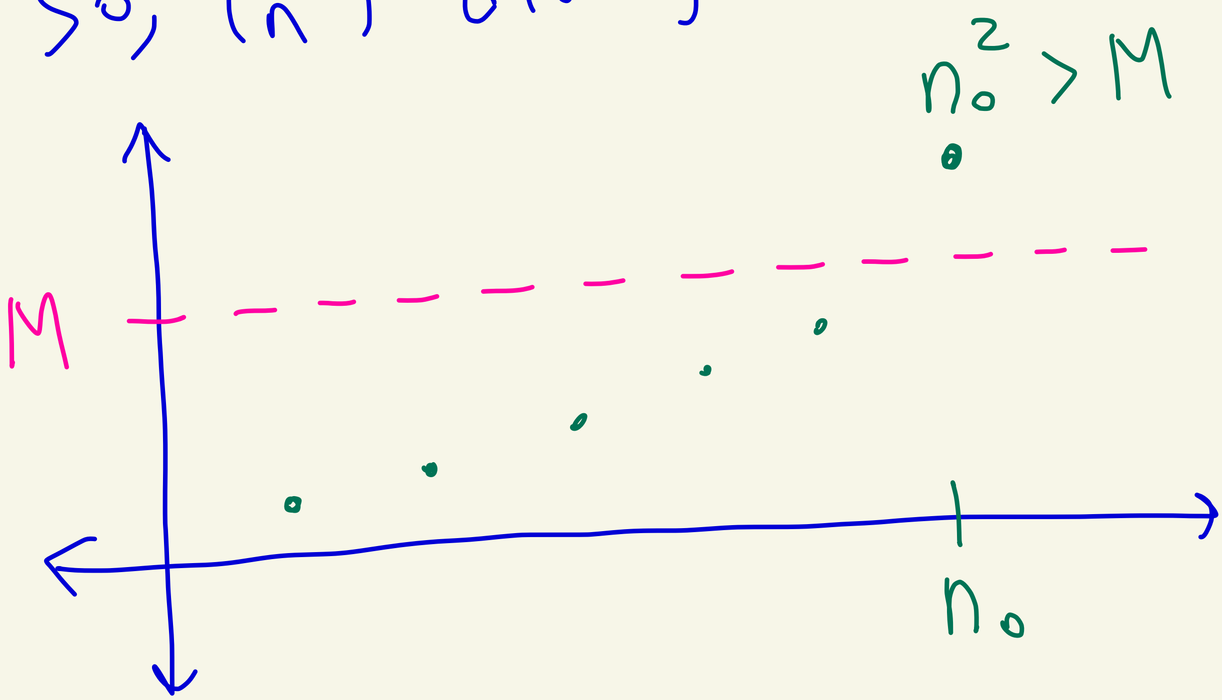
Let $M > 0$.

Pick some $n_0 > \sqrt{M}$.

Then, $|n_0^2| > M$.

So, (n^2) has no bound.

So, (n^2) diverges.



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Algebra of sequences theorem

Let (a_n) and (b_n) be convergent sequences with $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$. Let $\alpha \in \mathbb{R}$.

Then:

1. (αa_n) converges and $\lim_{n\to\infty} \alpha a_n = \alpha A$
2. $(a_n + b_n)$ converges and $\lim_{n\to\infty} (a_n + b_n) = A + B$
3. $(a_n b_n)$ converges and $\lim_{n\to\infty} a_n b_n = AB$
4. If $b_n \neq 0$ for all n and $B \neq 0$, then $\left(\frac{1}{b_n}\right)$ converges and $\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{B}$

Proof:

① If $\alpha=0$, then $\alpha a_n = 0$ for all n.

So,

$$\lim_{n\to\infty} \alpha a_n = \lim_{n\to\infty} 0 = 0 = 0\cdot A = \alpha A$$

last time

Now assume $\alpha \neq 0$.

Let $\varepsilon > 0$.

Since $\lim_{n\to\infty} a_n = A$ we know there exists N where if $n \geq N$ then $|a_n - A| < \frac{\varepsilon}{|\alpha|}$

So if $n \geq N$ then

$$|\alpha a_n - \alpha A| = |\alpha(a_n - A)|$$

$$= |\alpha|\,|a_n - A|$$

$$< |\alpha| \cdot \frac{\varepsilon}{|\alpha|}$$

$$= \varepsilon$$

So if $n \geq N$, then $|\alpha a_n - \alpha A| < \varepsilon$

Thus, $\lim\limits_{n \to \infty} \alpha a_n = \alpha A$.

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② Let $\varepsilon > 0$.

Note that

$$|(a_n + b_n) - (A+B)| = |(a_n - A) + (b_n - B)|$$

$$\underset{\uparrow \ \boxed{\triangle}}{\leq} |a_n - A| + |b_n - B|$$

Since $\lim\limits_{n \to \infty} a_n = A$ we know there

exists N_1 where if $n \geq N_1$
then $|a_n - A| < \varepsilon/2$.

Since $\lim_{n\to\infty} b_n = B$ we know there
exists N_2 where if $n \geq N_2$
then $|b_n - B| < \varepsilon/2$.

Let $N = \max\{N_1, N_2\}$
if $n \geq N$, then

from above

$$|(a_n + b_n) - (A+B)| \leq |a_n - A| + |b_n - B|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

So, $\lim(a_n + b_n) = A + B$.

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3 Let $\varepsilon > 0$.

Note that

$$|a_n b_n - AB| = |a_n b_n - b_n A + b_n A - AB|$$

$$\leq |a_n b_n - b_n A| + |b_n A - AB|$$

Δ-inequality

$$= |b_n||a_n - A| + |A||b_n - B|$$

Since (b_n) converges it is bounded.
So there exists $M > 0$ where

$$|b_n| \leq M \text{ for all } n.$$

Since $\lim_{n\to\infty} a_n = A$ there exists N_1
where if $n \geq N_1$ then $|a_n - A| < \frac{\varepsilon}{2M}$

Since $\lim_{n\to\infty} b_n = B$ there exists N_2
where if $n \geq N_2$ then $|b_n - B| < \frac{\varepsilon}{2(|A|+1)}$

Let $N = \max\{N_1, N_2\}$.

in case $|A| = 0$

Then, if $n \geq N$ we get

$$|a_n b_n - AB| \leq |b_n||a_n - A| + |A||b_n - B|$$

$$< M \cdot \frac{\varepsilon}{2M} + |A| \cdot \frac{\varepsilon}{2(|A|+1)}$$

$$= \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\underbrace{\left(\frac{|A|}{|A|+1}\right)}_{<1}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$= \varepsilon$$

So $\lim a_n b_n = AB$.

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4 Let $\varepsilon > 0$.

Note that

$$\left| \frac{1}{b_n} - \frac{1}{B} \right| = \left| \frac{B - b_n}{b_n B} \right|$$

$$= \frac{|B - b_n|}{|b_n B|}$$

$|-x| = |x|$

$$= \frac{|b_n - B|}{|b_n||B|}$$

Since $\lim_{n\to\infty} b_n = B$ there exists

N_1 where if $n \geq N_1$

then $|b_n - B| < \frac{|B|}{2}$.

So if $n \geq N_1$ then

$$\big| |b_n| - |B| \big| \leq |b_n - B| < \frac{|B|}{2}$$

HW

Thus if $n \geq N_1$ then

$$-\frac{|B|}{2} < |b_n| - |B| < \frac{|B|}{2}$$

$|x|<c$
iff
$-c<x<c$

So if $n \geq N_1$, then

$$\underbrace{\frac{|B|}{2} < |b_n|}_{\text{we want this}} < \frac{3|B|}{2}$$

Since $b_n \to B$ we can find N_2 where if $n \geq N_2$ then $|b_n - B| < \frac{\varepsilon}{2} \cdot |B|^2$

Let $N = \max\{N_1, N_2\}$.

If $n \geq N$, then

$$\left| \frac{1}{b_n} - \frac{1}{B} \right| \leq \frac{|b_n - B|}{|b_n|\,|B|}$$

$$= \frac{1}{|b_n|} \cdot \frac{1}{|B|} \cdot |b_n - B|$$

$$< \frac{2}{|B|} \cdot \frac{1}{|B|} \cdot \frac{\varepsilon}{2} \cdot |B|^2$$

$$= \varepsilon$$

So, $\lim\limits_{n \to \infty} \frac{1}{b_n} = \frac{1}{B}$ $\blacksquare$