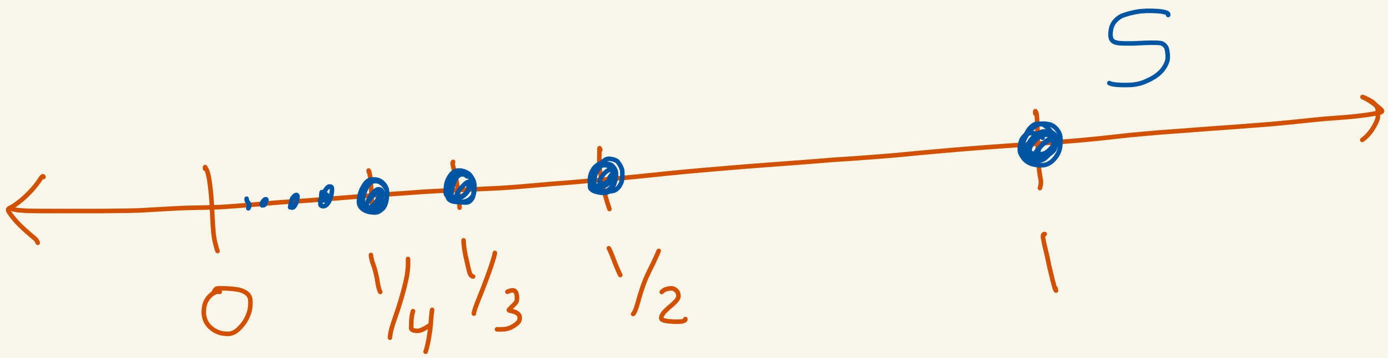
Math 4650
8/27/25

Ex: Let

$$S = \left\{ \frac{1}{n} \;\middle|\; n \in \mathbb{N} \right\}$$

$$= \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots \right\}$$



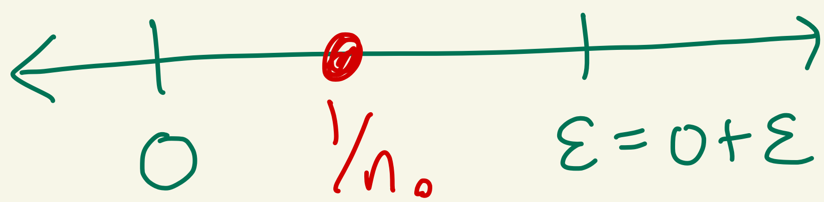
Let's show that $0 = \inf(S)$.

We know that 0 is a lower bound for S because

$$0 \leq \frac{1}{n} \text{ for all } n \in \mathbb{N}.$$

Let's use the Inf-Sup theorem.

Let $\varepsilon > 0$.



Pick some $n_0 \in \mathbb{N}$ where $n_0 > \frac{1}{\varepsilon}$

Then, $\frac{1}{n_0} < \varepsilon$

Then, $\frac{1}{n_0} \in S$ and $0 < \frac{1}{n_0} < 0 + \varepsilon$.

By the inf-sup theorem, $0 = \inf(S)$

$\blacksquare$

<u>Def:</u> Let $x \in \mathbb{R}$.

The <u>absolute value</u> of x is

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

<u>Ex:</u> $|5| = 5$

$|-3| = -(-3) = 3$

Theorem:

Let $a, b, c \in \mathbb{R}$ with $c > 0$.

Then:

① $|ab| = |a| \cdot |b|$

② $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ if $b \neq 0$

③ $|a| \leq c$ iff $-c \leq a \leq c$

④ $|a| < c$ iff $-c < a < c$

⑤ (triangle inequality)
$|a+b| \leq |a| + |b|$

⑥ $\big||a| - |b|\big| \leq |a-b|$

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proof: ①/② are in HW.

③

($\Rightarrow$) Suppose $|a| \le c$.

def of abs. value

assumption

If $a<0$, then $a < -a = |a| \le c$.

If $a \geq 0$, then $-a \le a = |a| \le c$.

In both cases we get

$a \le c$ and $-a \le c$.

So, $-c \le a \le c$.

($\Leftarrow$) Suppose $-c \le a \le c$.

Then, $-c \le a$ and $a \le c$.

So, $-a \le c$ and $a \le c$.

Thus, $|a| \le c$

since $|a| = a$ or $|a| = -a$

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④ Similar proof to ③ proof.

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5 Note first that if $x \in \mathbb{R}$ then $x \le |x|$.

Thus by taking $c = |x|$ and using part 3 we get

$$-|x| \le x \le |x|$$

3 says $|y| \le c$ iff $-c \le y \le c$

Thus if $a, b \in \mathbb{R}$, then

$$-|a| \le a \le |a| \quad \text{and} \quad -|b| \le b \le |b|$$

Adding gives

$$-(|a| + |b|) \le a + b \le |a| + |b|$$

Use part 3 again with $c = |a| + |b|$ to get:

$$|a + b| \le |a| + |b|.$$

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6 HW

Think of $|x-y|$ as the distance between x & y

We will use this alot:

Corollary:

Let $x, y, \varepsilon \in \mathbb{R}$ with $\varepsilon > 0$.

Then:

$$|x-y| < \varepsilon \quad \text{iff} \quad y-\varepsilon < x < y+\varepsilon$$

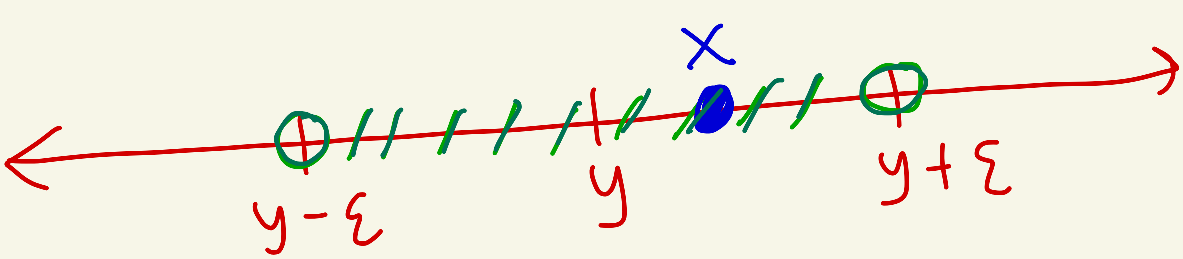
proof:

$$|x-y| < \varepsilon$$

part 3 above

$$\text{iff} \quad -\varepsilon < x-y < \varepsilon$$

$$\text{iff} \quad y-\varepsilon < x < y+\varepsilon$$



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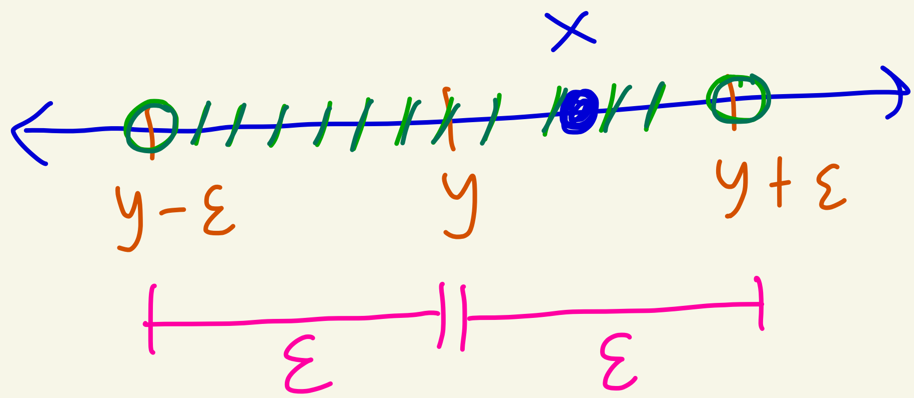
Some pictures:

$|x-y| < \varepsilon$



$\underbrace{0 < |x-y| < \varepsilon}_{\text{forces } x \neq y}$