Math 4650
8/20/25

<u>Def:</u> A <u>field</u> is a set F with two operations, addition and multiplication, such that:

(A1) If $x, y \in F$, then $x+y \in F$.

(A2) If $x, y \in F$, then $x+y = y+x$.

(A3) If $x, y, z \in F$, then

$$x+(y+z) = (x+y)+z$$

(A4) F contains an element 0 where $0+x = x$ for all $x \in F$.

(A5) For every $x \in F$ there exists an element $-x \in F$ where

$$x+(-x) = 0.$$

(M1) If $x, y \in F$, then $xy \in F$.

(M2) If $x, y \in F$, then $xy = yx$.

(M3) If $x, y, z \in F$, then

$$x(yz) = (xy)z.$$

(M4) F contains an element 1 where $1 \neq 0$ and $1x = x$ for all $x \in F$.

(M5) If $x \in F$ and $x \neq 0$ then there exists an element $x^{-1} \in F$ where $xx^{-1} = 1$.

(D1) If $x, y, z \in F$, then

$$x(y+z) = xy + xz$$

Def: An **ordered field** is a field F with a relation $<$ on F where

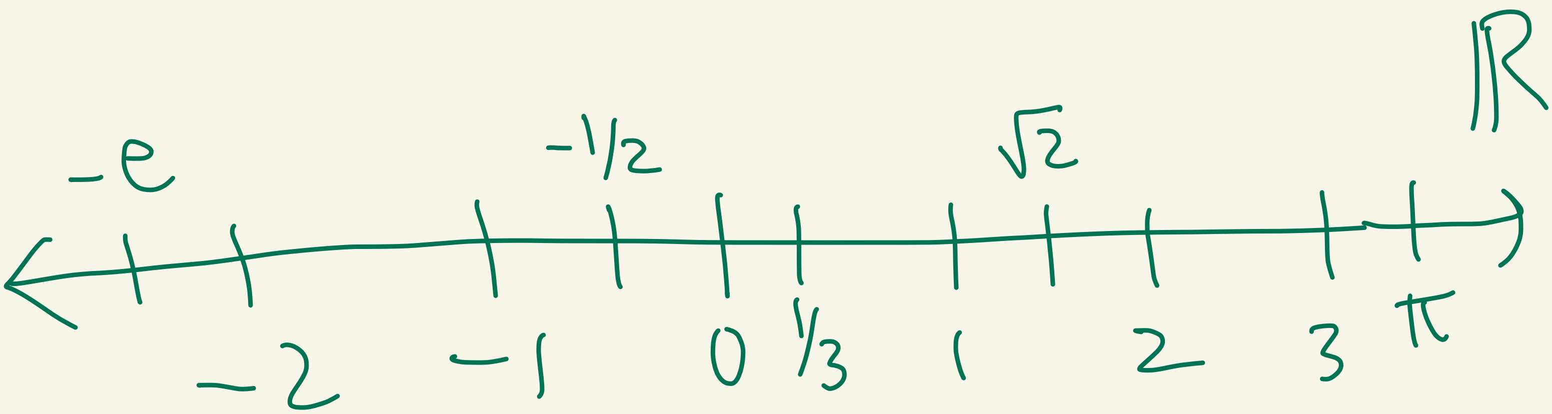
(O1) If $x, y \in F$, then one and only one of the following is true:
$x < y$ or $x = y$ or $y < x$

(O2) If $x, y, z \in F$ with $x < y$ and $y < z$, then $x < z$.

(O3) If $x, y, z \in F$ and $y < z$, then $x + y < x + z$.

(O4) If $x, y \in F$ with $x > 0$ and $y > 0$, then $xy > 0$

__Assumption:__ We will assume that the set of real numbers $\mathbb{R}$ exists and that it's an ordered field.



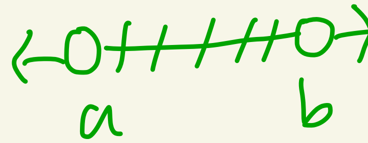
From the ordered field properties you can derive the other usual algebraic/order properties. We will assume all the other usual algebraic/order properties of $\mathbb{R}$.

See Topic 1a (optional) on how to do this

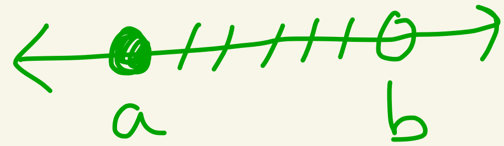
If there's time at the end of the semester I'll show how to construct $\mathbb{R}$ out of $\mathbb{Q}$ using "Dedekind cuts"

Interval notation

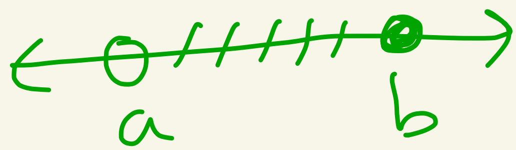
$(a,b) = \{x \in \mathbb{R} \mid a < x < b\}$



$[a,b) = \{x \in \mathbb{R} \mid a \le x < b\}$



$(a,b] = \{x \in \mathbb{R} \mid a < x \le b\}$



$[a,b] = \{x \in \mathbb{R} \mid a \le x \le b\}$

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natural numbers:

$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$

integers:

$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

rationals:

$\mathbb{Q} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0 \right\}$