

Math 4650

8/20/25

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Def: A field is a set  $F$  with two operations, addition and multiplication, such that:

(A1) If  $x, y \in F$ , then  $x + y \in F$ .

(A2) If  $x, y \in F$ , then  $x + y = y + x$ .

(A3) If  $x, y, z \in F$ , then  
$$x + (y + z) = (x + y) + z$$

(A4)  $F$  contains an element  $0$   
where  $0 + x = x$  for all  $x \in F$ .

(A5) For every  $x \in F$  there exists  
an element  $-x \in F$  where  
$$x + (-x) = 0.$$

(M1) If  $x, y \in F$ , then  $xy \in F$ .

(M2) If  $x, y \in F$ , then  $xy = yx$ .

(M3) If  $x, y, z \in F$ , then  
$$x(yz) = (xy)z.$$

(M4)  $F$  contains an element  $1$  where  
 $1 \neq 0$  and  $1x = x$   
for all  $x \in F$ .

(M5) If  $x \in F$  and  $x \neq 0$  then  
there exists an element  
 $x^{-1} \in F$  where  $xx^{-1} = 1$ .

(D1) If  $x, y, z \in F$ , then  
$$x(y+z) = xy + xz$$

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Def: An ordered field is a field  $F$  with a relation  $<$  on  $F$  where

(01) If  $x, y \in F$ , then one and only one of the following is true:

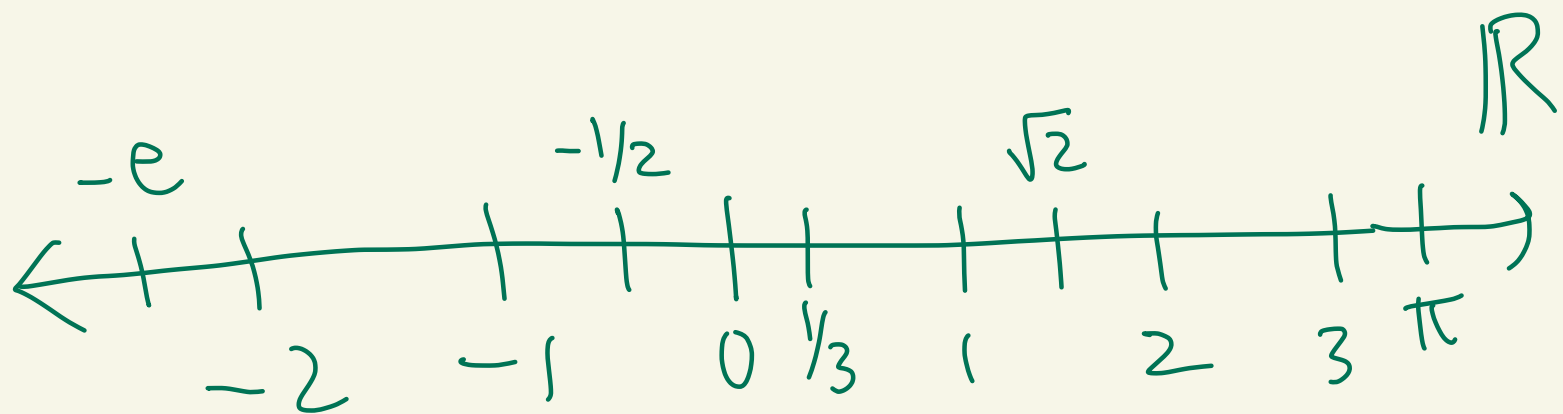
$$x < y \text{ or } x = y \text{ or } y < x$$

(02) If  $x, y, z \in F$  with  $x < y$  and  $y < z$ , then  $x < z$ .

(03) If  $x, y, z \in F$  and  $y < z$ , then  $x + y < x + z$ .

(04) If  $x, y \in F$  with  $x > 0$  and  $y > 0$ , then  $xy > 0$

Assumption: We will assume that the set of real numbers  $\mathbb{R}$  exists and that it's an ordered field.



From the ordered field properties you can derive the other usual algebraic/order properties. We will assume all the other usual algebraic/order properties of  $\mathbb{R}$ .

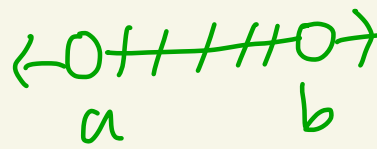
See Topic 1a (optional)  
on how to do this

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If there's time at the  
end of the semester I'll  
show how to construct  
 $\mathbb{R}$  out of  $\mathbb{Q}$  using  
"Dedekind cuts"

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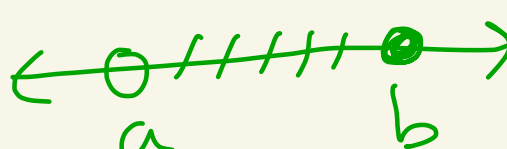
## Interval notation

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$$



A horizontal number line with arrows at both ends. Two points are marked with open circles (o) and labeled 'a' and 'b' below them. The segment between 'a' and 'b' is filled with several parallel diagonal hash marks (//).

$$[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$$


A horizontal number line with arrows at both ends. Two points are marked: 'a' with a closed circle (●) and 'b' with an open circle (o). The segment between 'a' and 'b' is filled with several parallel diagonal hash marks (//).

$$(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$$


A horizontal number line with arrows at both ends. Two points are marked: 'a' with an open circle (o) and 'b' with a closed circle (●). The segment between 'a' and 'b' is filled with several parallel diagonal hash marks (//).

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$$


A horizontal number line with arrows at both ends. Two points are marked with closed circles (●) and labeled 'a' and 'b' below them. The segment between 'a' and 'b' is filled with several parallel diagonal hash marks (//).

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natural numbers:

$$\mathbb{N} = \{1, 2, 3, 4, 5, \dots\}$$

integers:

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$$

rational numbers:

$$\mathbb{Q} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0 \right\}$$