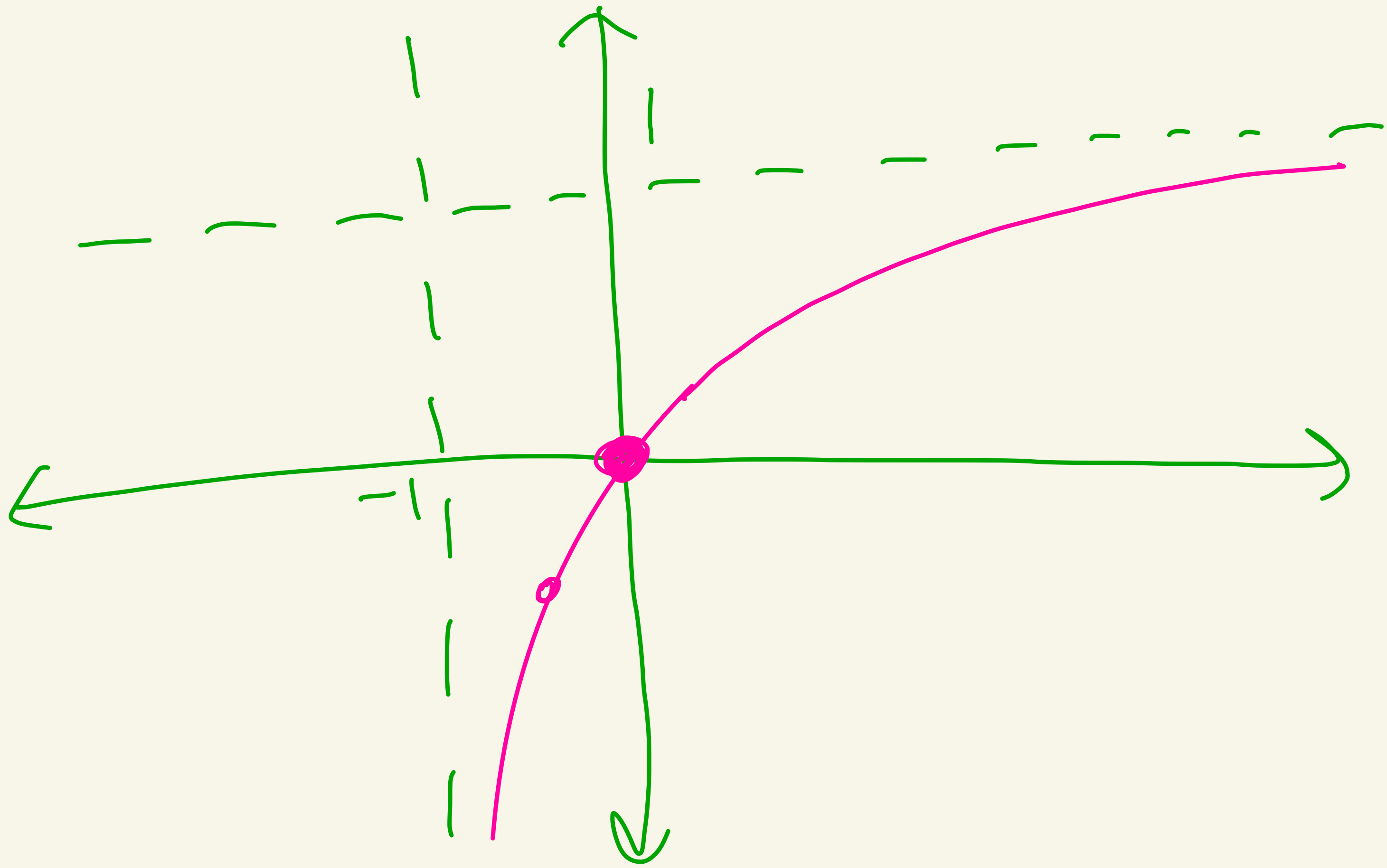
Math 4650

12/3/25

$$X = \left\{ \frac{x}{1+x} \;\middle|\; x \in \mathbb{R} \text{ and } x > -1 \right\}$$



$\inf(X) \leftarrow$ DNE

$\sup(X) = 1$

Test 1

E $a_n \to A$, $b_n \to B$

$\alpha, \beta \neq 0$

prove: $\alpha a_n + \beta b_n \to \alpha A + \beta B$

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proof:

Let $\varepsilon > 0$.

Note that

$$|(\alpha a_n + \beta b_n) - (\alpha A + \beta B)|$$

$$= |\alpha a_n - \alpha A + \beta b_n - \beta B|$$

$$\leq |\alpha a_n - \alpha A| + |\beta b_n - \beta B|$$

$$= |\alpha||a_n - A| + |\beta||b_n - B|$$

Since $\lim_{n\to\infty} a_n = A$ there exists

N_1 where if $n \geq N_1$ then

$$|a_n - A| < \frac{\varepsilon}{2|\alpha|}$$

Since $\lim_{n\to\infty} b_n = B$ there exists

N_2 where if $n \geq N_2$ then

$$|b_n - B| < \frac{\varepsilon}{2|\beta|}.$$

Let $N = \max\{N_1, N_2\}$ ← So, $N \geq N_1$ and $N \geq N_2$

Then if $n \geq N$, we get

$$|(\alpha a_n + \beta b_n) - (\alpha A + \beta B)|$$

$$\leq |\alpha||a_n - A| + |\beta||b_n - B|$$

$$< |\alpha| \cdot \frac{\varepsilon}{2|\alpha|} + |\beta| \cdot \frac{\varepsilon}{2|\beta|}$$

$$= \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$= \varepsilon. \qquad \blacksquare$$

$\boxed{\text{HW 2}}$

$\boxed{\#2(e)}$

$$\lim_{n \to \infty} \frac{n^2}{2n^2+1} = \frac{1}{2}$$

Proof: Let $\varepsilon > 0$.

Note that

$$\left| \frac{n^2}{2n^2+1} - \frac{1}{2} \right| = \left| \frac{2n^2 - 2n^2 - 1}{2(2n^2+1)} \right|$$

$$= \left| \frac{-1}{4n^2+2} \right| = \frac{1}{4n^2+2}$$

We have

$$\frac{1}{4n^2+2} < \varepsilon$$

iff

$$\frac{1}{\varepsilon} < 4n^2 + 2$$

iff

$$\frac{1}{4\varepsilon} - \frac{1}{2} < n^2$$

iff

$$\sqrt{\frac{1}{4\varepsilon} - \frac{1}{2}} < n$$

Pick $N > \sqrt{\frac{1}{4\varepsilon} - \frac{1}{2}}$.

Then if $n \geq N$, we have

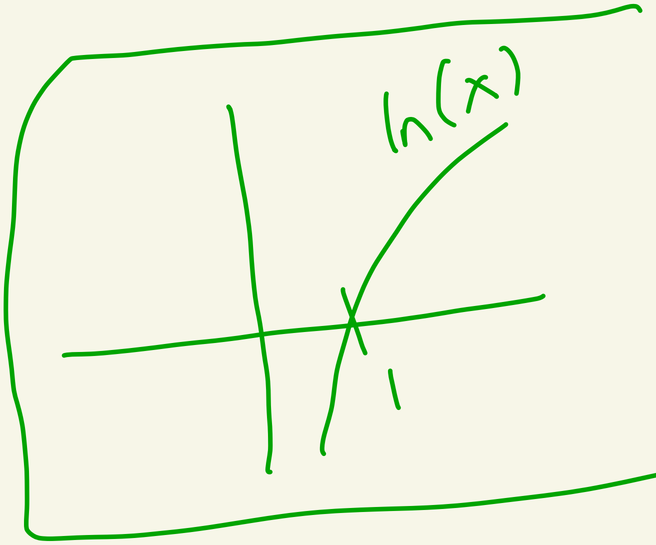
$$\left| \frac{n^2}{2n^2+1} - \frac{1}{2} \right| < \varepsilon \qquad \blacksquare$$

$\boxed{\text{HW 2}}$ – (3)(f) $\lim_{n\to\infty} r^n = 0$ if $0<r<1$

Let $\varepsilon > 0$.

Then,

$$|r^n - 0| = |r^n| = r^n$$



Note that

$$r^n < \varepsilon$$

iff

$$\ln(r^n) < \ln(\varepsilon)$$

iff

$$n \underbrace{\ln(r)}_{\text{negative}} < \underbrace{\ln(\varepsilon)}_{\substack{\text{negative} \\ \text{assume } \varepsilon<1}}$$

iff

$$n > \frac{\ln(\varepsilon)}{\ln(r)}$$

Pick $N > \dfrac{\ln(\varepsilon)}{\ln(r)}$.

If $n \geq N$, then $|r^n - 0| < \varepsilon$

$\blacksquare$

$$\sum_{n=1}^{\infty} (-1)^n \frac{2n}{3n+2}$$

$$\lim_{n\to\infty} \left| (-1)^n \frac{2n}{3n+2} \right| = \lim_{n\to\infty} \overbrace{\left|(-1)^n\right|}^{1} \left| \frac{2n}{3n+2} \right|$$

$$= \lim_{n\to\infty} \frac{2n}{3n+2}$$

$$= \lim_{n\to\infty} \frac{2n/n}{3n/n + 2/n}$$

$$= \lim_{n\to\infty} \frac{2}{3+2/n} = \frac{2}{3+0}$$

$$= \frac{2}{3} \neq 0$$

So, $\lim\limits_{n\to\infty} (-1)^n \frac{2n}{3n+2} \neq 0$.

$\left[\lim\limits_{n\to\infty} |a_n| = 0 \text{ iff } \lim\limits_{n\to\infty} a_n = 0 \right]$

So, $\sum\limits_{n=1}^{\infty} (-1)^n \frac{2n}{3n+2}$ diverges.

Ex: Show that

$$\lim_{x \to 2} x^3 = 8$$

Hint: $|x^3 - 8| = |(x-2)(x^2+2x+4)|$

Proof:

Let $\varepsilon > 0$.

Note that

$$|x^3 - 8| = \underbrace{|x-2|}_{< \delta} \underbrace{|x^2+2x+4|}_{\text{need to bound}}$$

$$\boxed{\begin{array}{c} |x| < c \\ \text{iff} \\ -c < x < c \end{array}}$$

Suppose $\delta \leq 1$.

Suppose $0 < |x-2| < \delta \leq 1$

So, $-1 < x - 2 < 1$

Then, $1 < x < 3$.

So, $|x|<3$.

Thus, $|x^2+2x+4| \le |x^2|+|2x|+|4|$

$$= |x|^2 + 2|x| + 4$$

$$< 3^2 + 2\cdot 3 + 4$$

$$= 19.$$

So, if $0<|x-2|<\delta\le 1$, then

$$|x^3-8| = |x-2||x^2+2x+4|$$

$$< |x-2|\cdot 19$$

Let $\delta = \min\left\{1, \frac{\varepsilon}{19}\right\}$.

So, $\delta \le 1$ and $\delta \le \frac{\varepsilon}{19}$.

Then if $0<|x-2|<\delta$, then

$$\underset{\boxed{\delta\le 1}}{|x^3-8| < |x-2|\cdot 19} < \delta\cdot 19$$

$$\le \frac{\varepsilon}{19}\cdot 19$$

$$= \varepsilon. \quad \blacksquare$$