Math 4650

11/5/25

Schedule

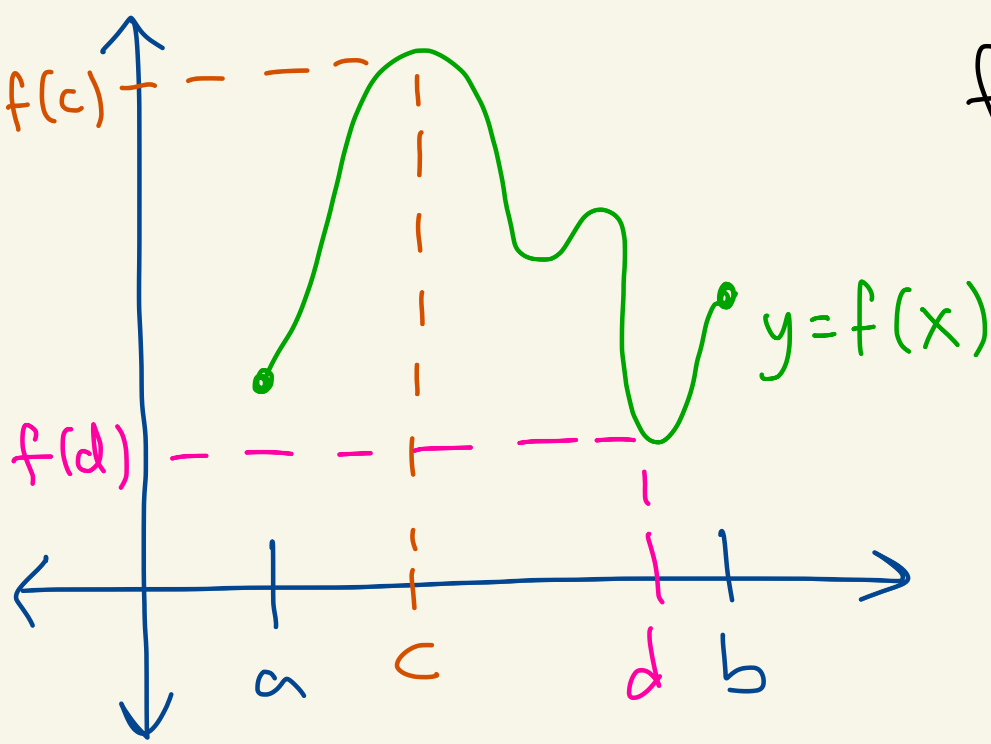
	11/5 Topic 5
11/10 Review	11/12 Review
11/17 Review	11/19 Test 2
11/24 HOLIDAY	11/26 HOLIDAY
12/1 Topic 6	12/3 Topic 6
12/8 Final 2:30-4:30	

Theorem: Let f be continuous on $[a,b]$ with $a<b$.

Then f attains it's maximum and minimum on $[a,b]$.

That is, there exists c with $a \leq c \leq b$ with $f(c) \geq f(x)$ for all $a \leq x \leq b$.

And there exists d with $a \leq d \leq b$ with $f(d) \leq f(x)$ for all $a \leq x \leq b$.



proof:

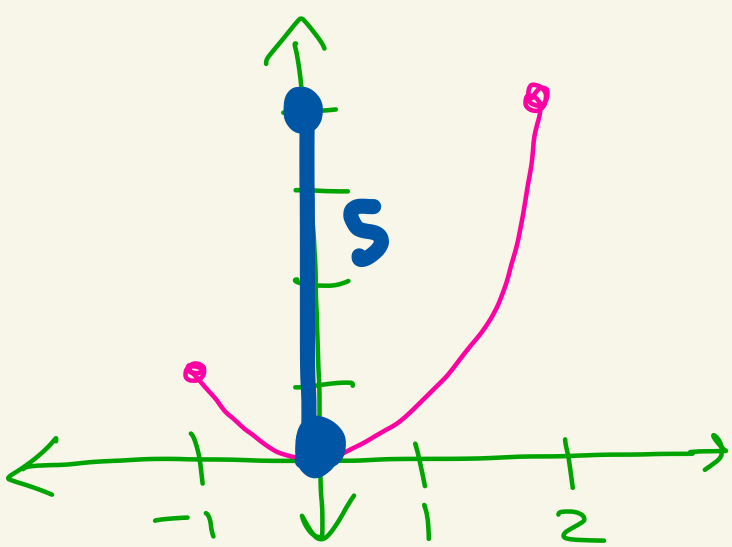
We will prove that f attains it's maximum. For the minimum, just repeat the proof with $-f$ instead of f.

Consider

$$S = \{ f(x) \mid a \leq x \leq b \}$$

Ex: $f(x) = x^2$

$[a,b] = [-1,2]$



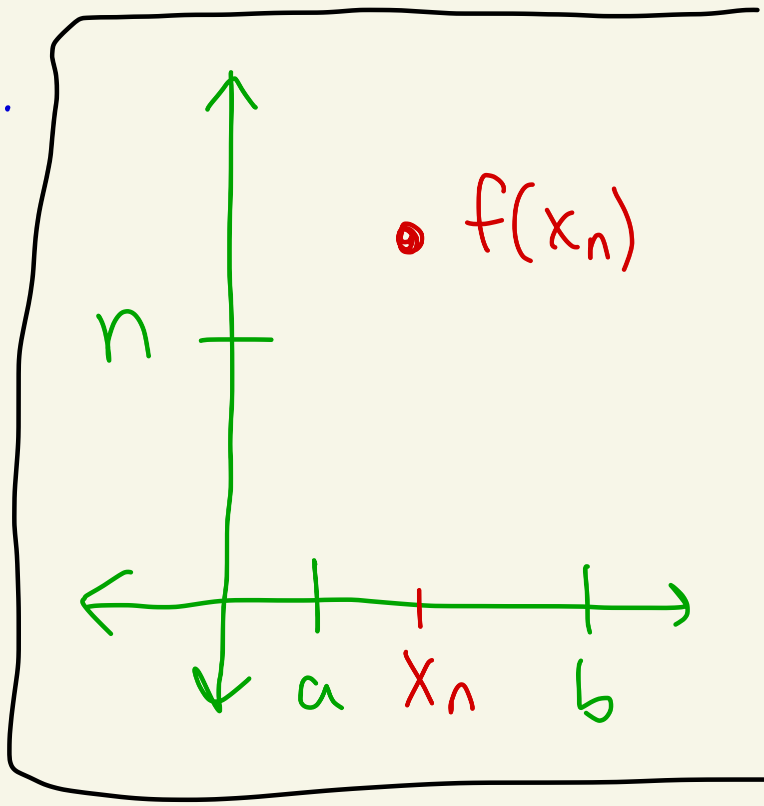
$S = [0,4]$

Let's show that S is bounded from above.

Suppose S is not bounded from above.

Then for each $n \in \mathbb{N}$ there exists $a \leq x_n \leq b$ where

$$f(x_n) > n.$$



Since $a \leq x_n \leq b$ for all n, the sequence $(x_n)_{n=1}^{\infty}$ is a bounded sequence.

The Bolzano-Weierstrass theorem tells us that we get a convergent subsequence

$x_{n_1}, x_{n_2}, x_{n_3}, \ldots$

with $n_1 < n_2 < n_3 < \cdots$

Suppose $\lim\limits_{n_k \to \infty} x_{n_k} = c$.

Since $a \le x_{n_k} \le b$,

by HW 2 #6(c)

we get $a \le \lim\limits_{n_k \to \infty} x_{n_k} \le b$.

So, $a \le c \le b$.

Since f is continuous at c,

$$f(c) = \lim_{n_k \to \infty} f(x_{n_k})$$

HW 5 #2

But the limit on the right doesn't exist because $f(x_{n_k}) \geq n_k$

so $f(x_{n_k}) \longrightarrow \infty$

because $n_k \to \infty$.

Contradiction.

So, S is bounded!

Thus, $M = \sup(S)$ exists.

By the inf/sup theorem, for each $m \in \mathbb{N}$ there exist y_m with $a \leq y_m \leq b$

with
$$M - \frac{1}{m} < f(y_m) \leq M$$

So, $M - 1 < f(y_1) \le M$

$$M - \frac{1}{2} < f(y_2) \le M$$

$$M - \frac{1}{3} < f(y_3) \le M$$

$$M - \frac{1}{4} < f(y_4) \le M$$

$$\vdots \qquad \vdots \qquad \vdots$$

Here $\lim\limits_{m \to \infty} f(y_m) = M$.

Since $(y_m)_{m=1}^{\infty}$ is a bounded sequence $[a \le y_m \le b, \forall m]$

there exists a convergent subsequence $(y_{m_\ell})_{\ell=1}^{\infty}$

Suppose $\lim\limits_{m_\ell \to \infty} y_{m_\ell} = d$.

Since $a \le y_{m_\ell} \le b$ for all m_ℓ we get $a \le d \le b$.

Since f is continuous at d,

$$f(d) = \lim_{m_\ell \to \infty} f(y_{m_\ell}) = M$$

HW 5 #2

$\lim_{m \to \infty} f(y_m) = M$ HW 2

Since $f(d) = M = \sup(S)$ we know $f(d) \ge f(x)$ for all $a \le x \le b$.

Thus, f attains its maximum at d. $\blacksquare$