Math 4650

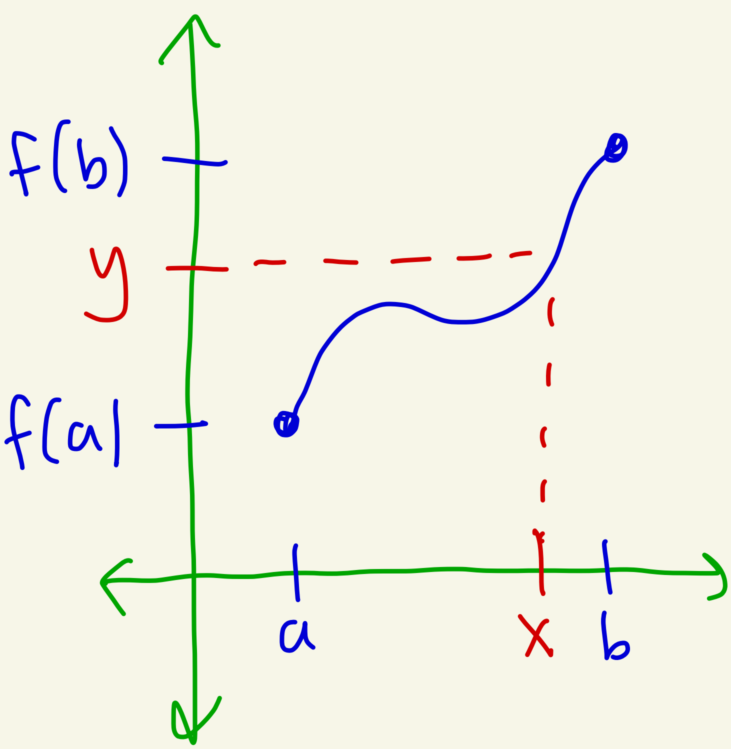
11/3/25

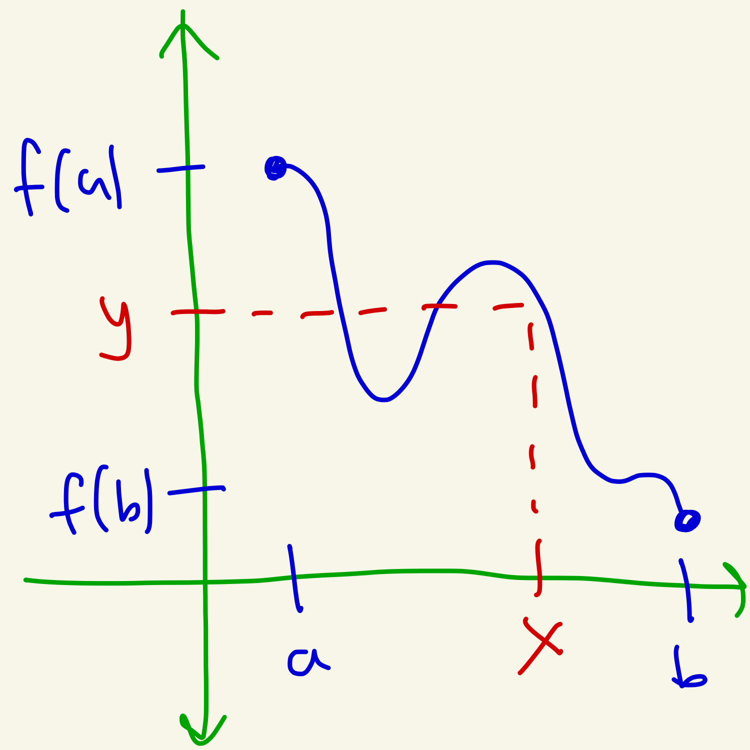
Theorem (Intermediate value theorem)

Let f be continuous on $[a,b]$ with $a<b$.
If for some $y \in \mathbb{R}$ we have either

$f(a) < y < f(b)$ or $f(b) < y < f(a)$

then there exists x with $a < x < b$ with $f(x) = y$.





Proof:

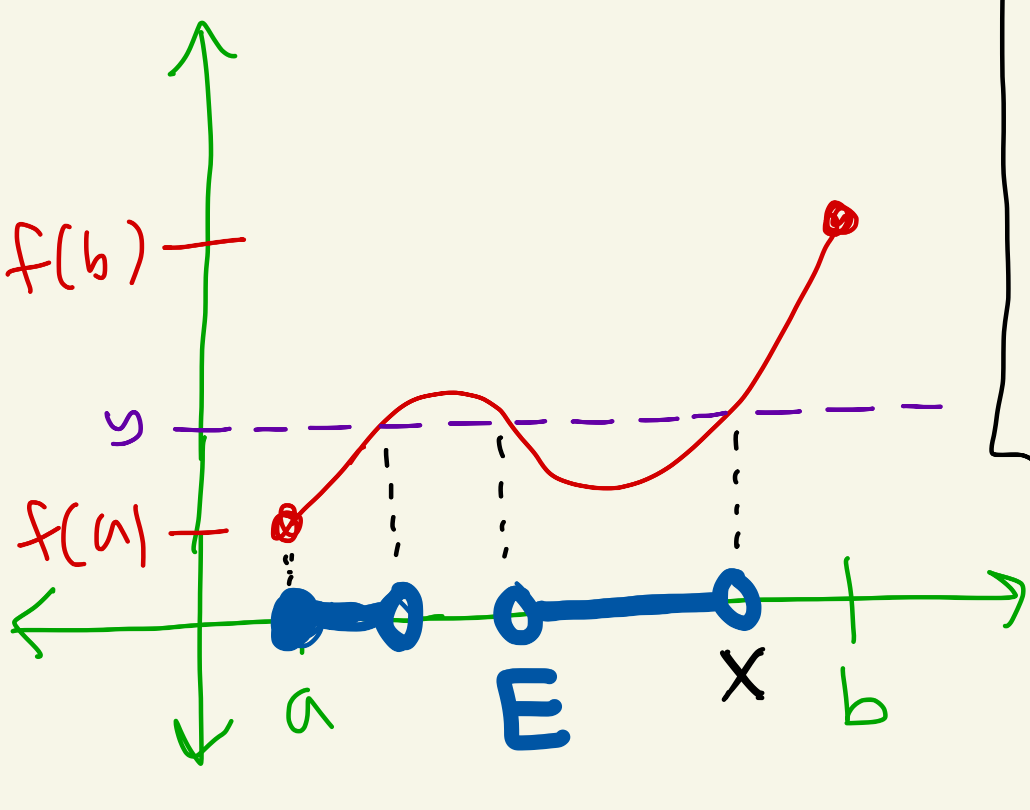
We will prove the case when

$$f(a) < y < f(b)$$

The other case is similar.

Define

$$E = \{ t \mid a \leq t \leq b \text{ and } f(t) < y \}$$



Note that $a \in E$ because $a \leq a \leq b$ and $f(a) < y$.

So, $E \neq \emptyset$

And $E \subseteq [a,b]$

So, b is an upper bound for E.

By the completeness axiom, $x = \sup(E)$ exists.

$$\boxed{\begin{array}{l}\text{We will prove:} \\ \quad a < x < b \text{ and } f(x) = y\end{array}}$$

From HW 1, if $A \subseteq B$, then

$$\inf(B) \leq \inf(A) \leq \sup(A) \leq \sup(B).$$

Since $E \subseteq [a,b]$ we know

$$a \leq \inf(E) \leq \underbrace{\sup(E)}_{x} \leq b.$$

So, $a \leq x \leq b$.

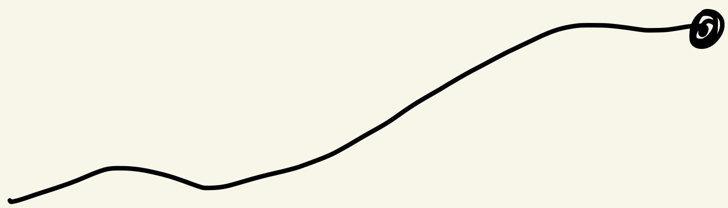
Let's show that $a < x < b$.

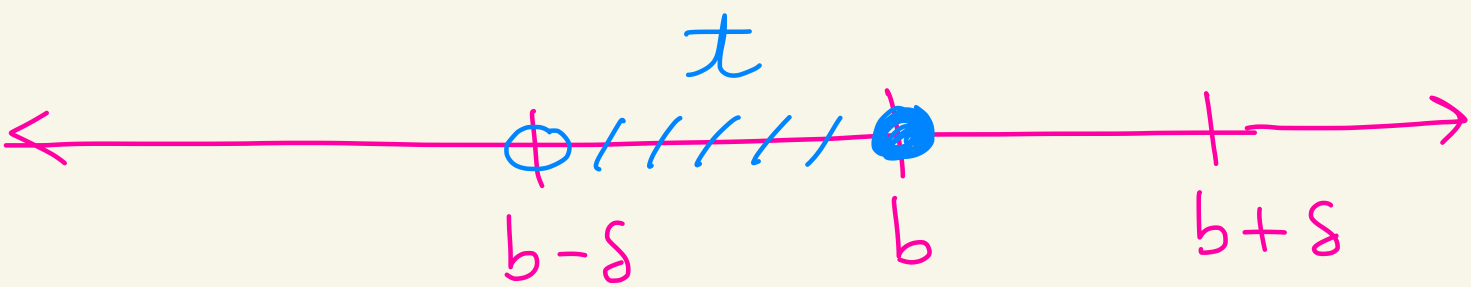
First let's show $x < b$.

We know $y < f(b)$.

Since f is continuous at b,
there exists $\delta > 0$ so that
if $t \in [a, b]$ and $|t - b| < \delta$
then $|f(t) - f(b)| < \underbrace{f(b) - y}$

$\varepsilon = f(b) - y$
$\varepsilon > 0$ since $y < f(b)$





So if $b - \delta < t \leq b$, then

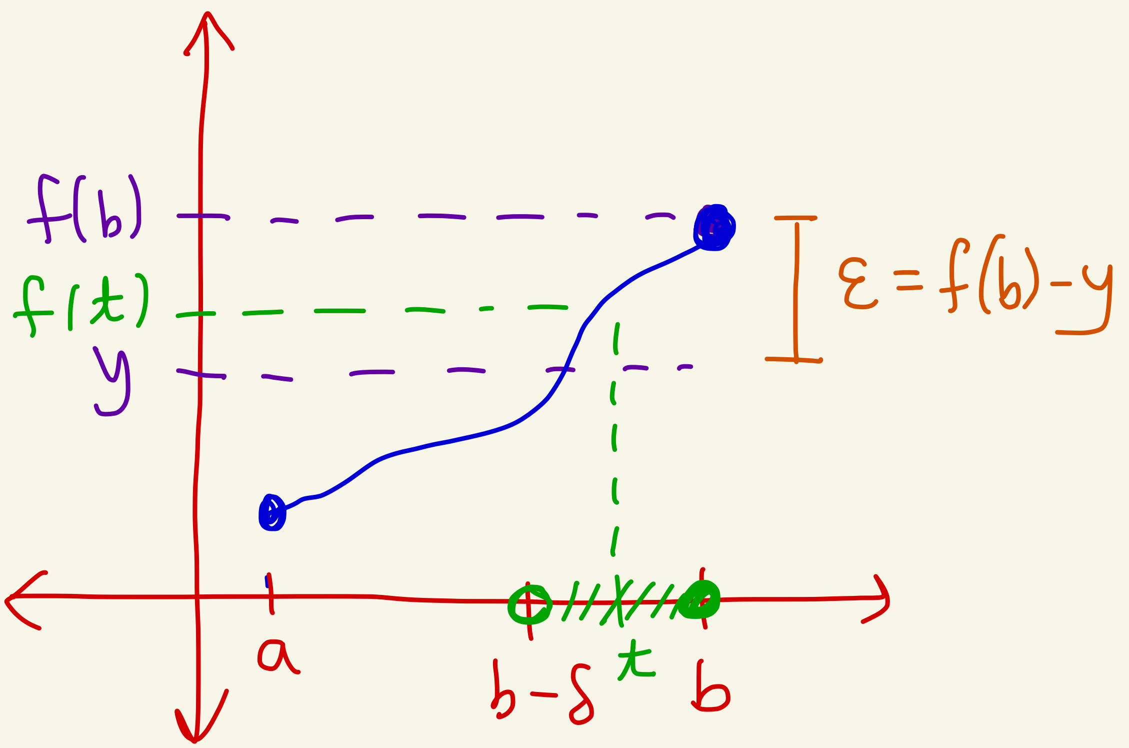
$$-(f(b) - y) < f(t) - f(b) < f(b) - y$$

So if $b - \delta < t \leq b$, then

$$y < f(t) < 2f(b) - y$$

So if $b-\delta < t \le b$,

then $y < f(t)$.



Since $E = \{t \mid a \le t \le b \text{ and } f(t) < y\}$

we know $E \cap (b-\delta, b] = \emptyset$.

So, $b-\delta$ is an upper bound for E.

Thus,

$x = \sup(E) \le b - \delta < b$

$\boxed{\delta > 0}$

So, $x < b$.

Similarly you can show $a < x$.

Thus, $a < x < b$.

Now we show $f(x) = y$.

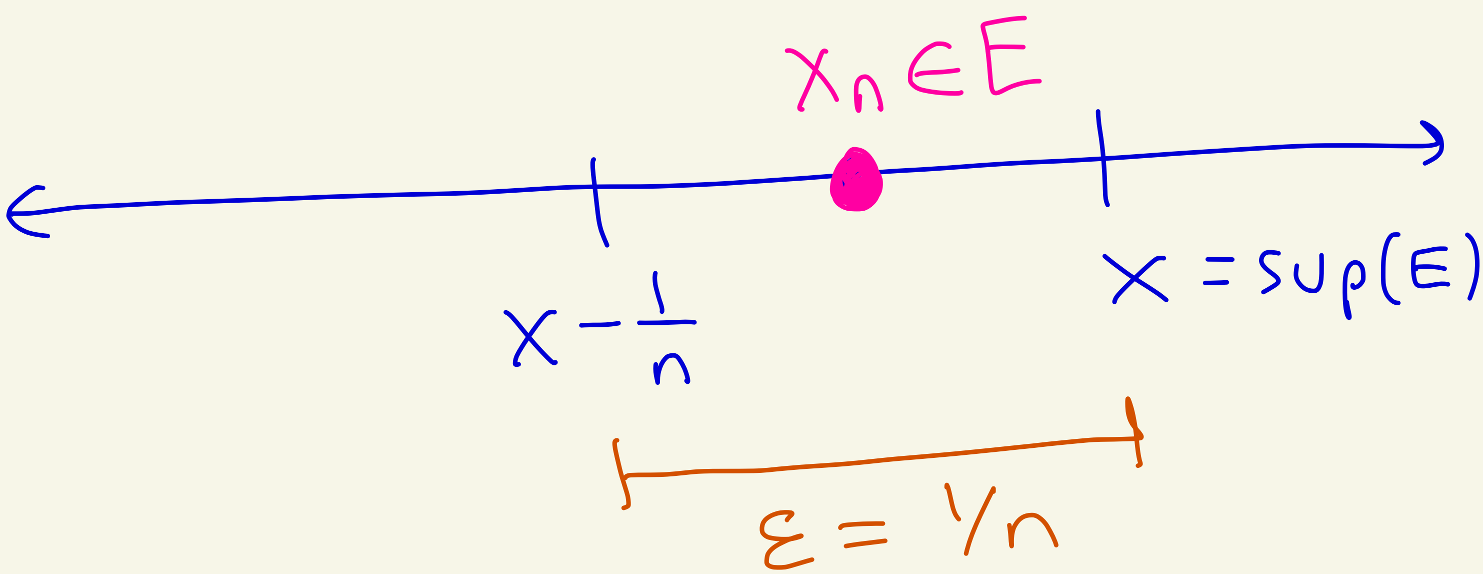
First we show $f(x) \leq y$.

By the inf-sup theorem,

since $x = \sup(E)$,

for each $n \in \mathbb{N}$ there exists

$x_n \in E$ with $x - \frac{1}{n} < x_n \leq x$



This gives a sequence

$x_1, x_2, x_3, x_4, \ldots$

of points from E

with $\lim_{n \to \infty} x_n = x$.

Since each $x_n \in E$ we know $f(x_n) < y$.

Since f is continuous at x, and $x \in [a,b]$ and $\lim_{n\to\infty} x_n = x$, by the previous theorem

we know that

$$\lim_{n \to \infty} f(x_n) = f(x)$$

By HW 2, since $f(x_n) < y$ for all n, we get

$$\lim_{n \to \infty} f(x_n) \leq \lim_{n \to \infty} y$$

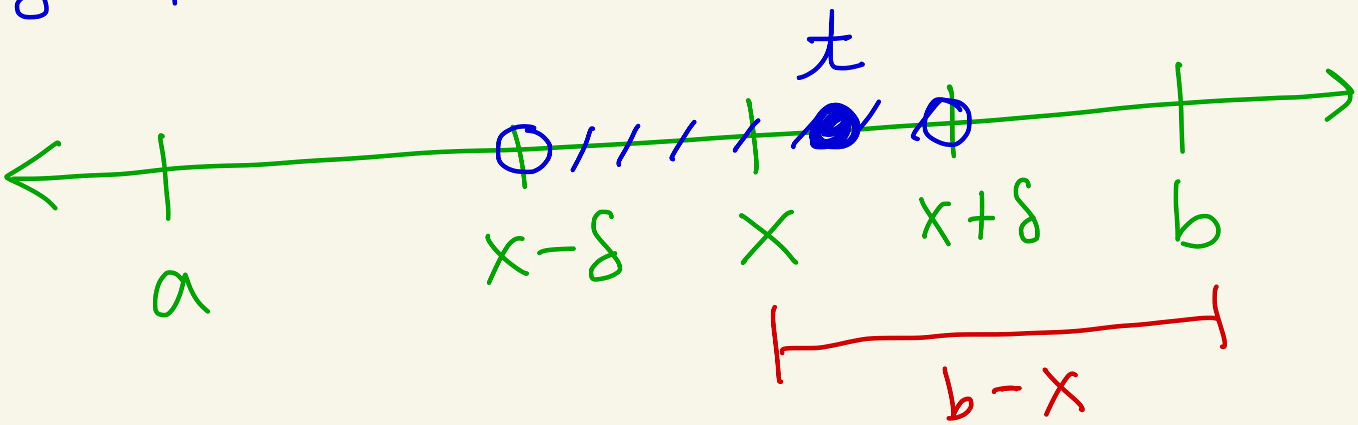
So, $f(x) \leq y$.

Now let's show that $f(x) \geq y$.
We will rule out $f(x) < y$.
Suppose $f(x) < y$.
Since f is continuous at x there exists $\delta > 0$ where if $t \in [a,b]$ and $|t - x| < \delta$, then

$$|f(t) - f(x)| < \underbrace{y - f(x)}$$

$\varepsilon = y - f(x) > 0$ because $f(x) < y$

In the above we can assume that $\delta < b - x$ by shrinking δ if needed.



So if $x - \delta < t < x + \delta < b$

then

$$-(y - f(x)) < f(t) - f(x) < y - f(x)$$

So if $x - \delta < t < x + \delta < b$, then $-y + 2f(x) < f(t) < y$.

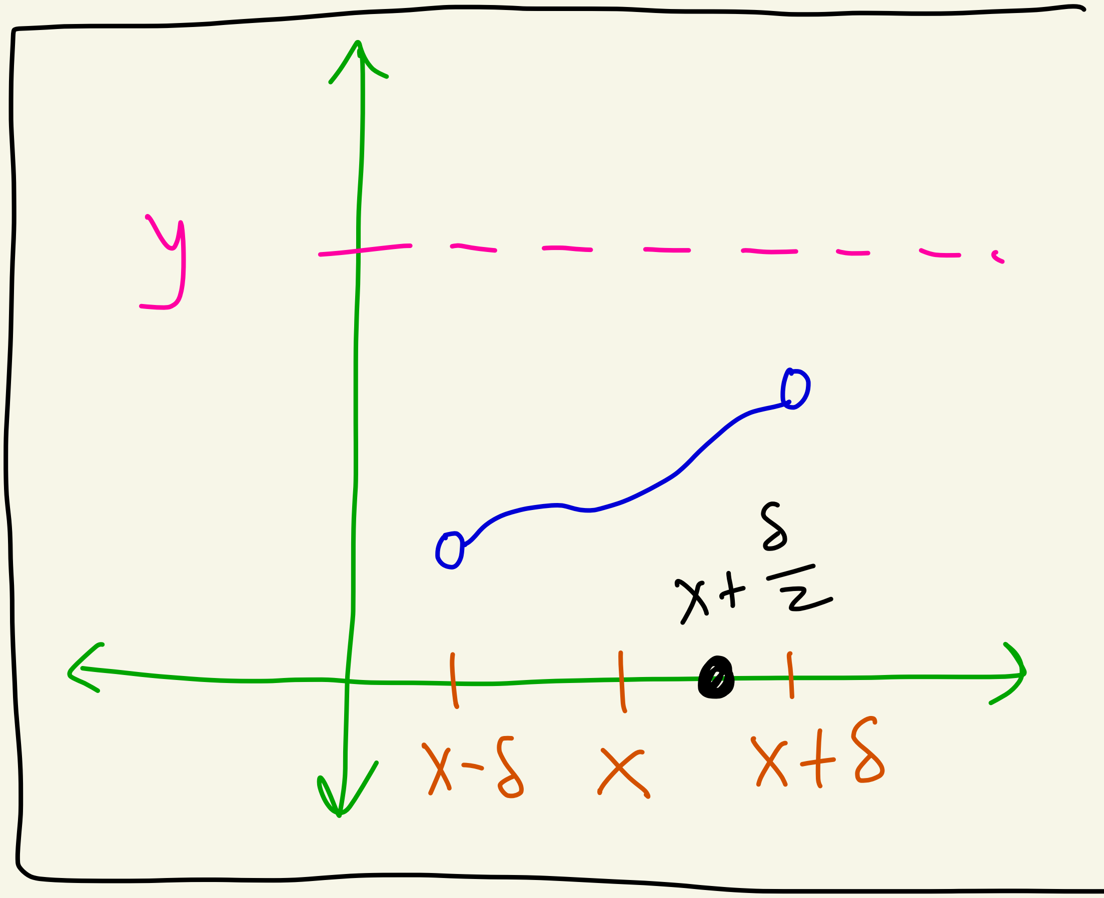
So if $x - \delta < t < x + \delta < b$

then $f(t) < y$.

Then, $(x-\delta, x+\delta) \subseteq E$

But then $x + \frac{\delta}{2} \in E$

This contradicts that $x = \sup(E)$.



Thus, $f(x) < y$ is impossible.

Therefore, $f(x) \geq y$.

Since $f(x) \leq y$ and $f(x) \geq y$ we get $f(x) = y$. ▨

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