Math 4650

11/17/25

HW 3 - 8(a)

Prove:

If $\sum_{n=1}^{\infty} |a_n|$ converges,

then $\sum_{n=1}^{\infty} a_n$ converges

Proof:

Let $s_k = |a_1| + |a_2| + \dots + |a_k|$

and $t_k = a_1 + a_2 + \dots + a_k$.

We are proving:

If (s_k) converges,

then (t_k) converges.

Instead we prove:

If (s_k) is Cauchy,

then (t_k) is Cauchy.

Let $\varepsilon > 0$.

Since (s_k) is Cauchy there exists $N > 0$ where if $m \geq n \geq N$, then $|s_m - s_n| < \varepsilon$

So, if $m \geq n \geq N$, then

$$\varepsilon > |s_m - s_n|$$

$$= \big|\left(|a_1| + |a_2| + \dots + |a_m|\right) - \left(|a_1| + |a_2| + \dots + |a_n|\right)\big|$$

$$= \big|\, |a_{n+1}| + |a_{n+2}| + \dots + |a_m| \,\big|$$

$$= |a_{n+1}| + |a_{n+2}| + \dots + |a_m|$$

$$\overset{\triangle}{\geq} |a_{n+1} + a_{n+2} + \dots + a_m|$$

$$= |(a_1 + a_2 + \dots + a_m) - (a_1 + a_2 + \dots + a_n)|$$

$$= |t_m - t_n|$$

So, if $m \geq n \geq N$, then

$$|t_m - t_n| < \varepsilon.$$

Thus (t_k) is Cauchy. $\blacksquare$

HW 5

① (d) Show that $f(x) = \frac{1}{x^2}$ is continuous at $a > 0$.

NTS: $\lim\limits_{x \to a} \frac{1}{x^2} = \frac{1}{a^2}$

We can control $|x-a|$ by δ

proof:

Let $\varepsilon > 0$.

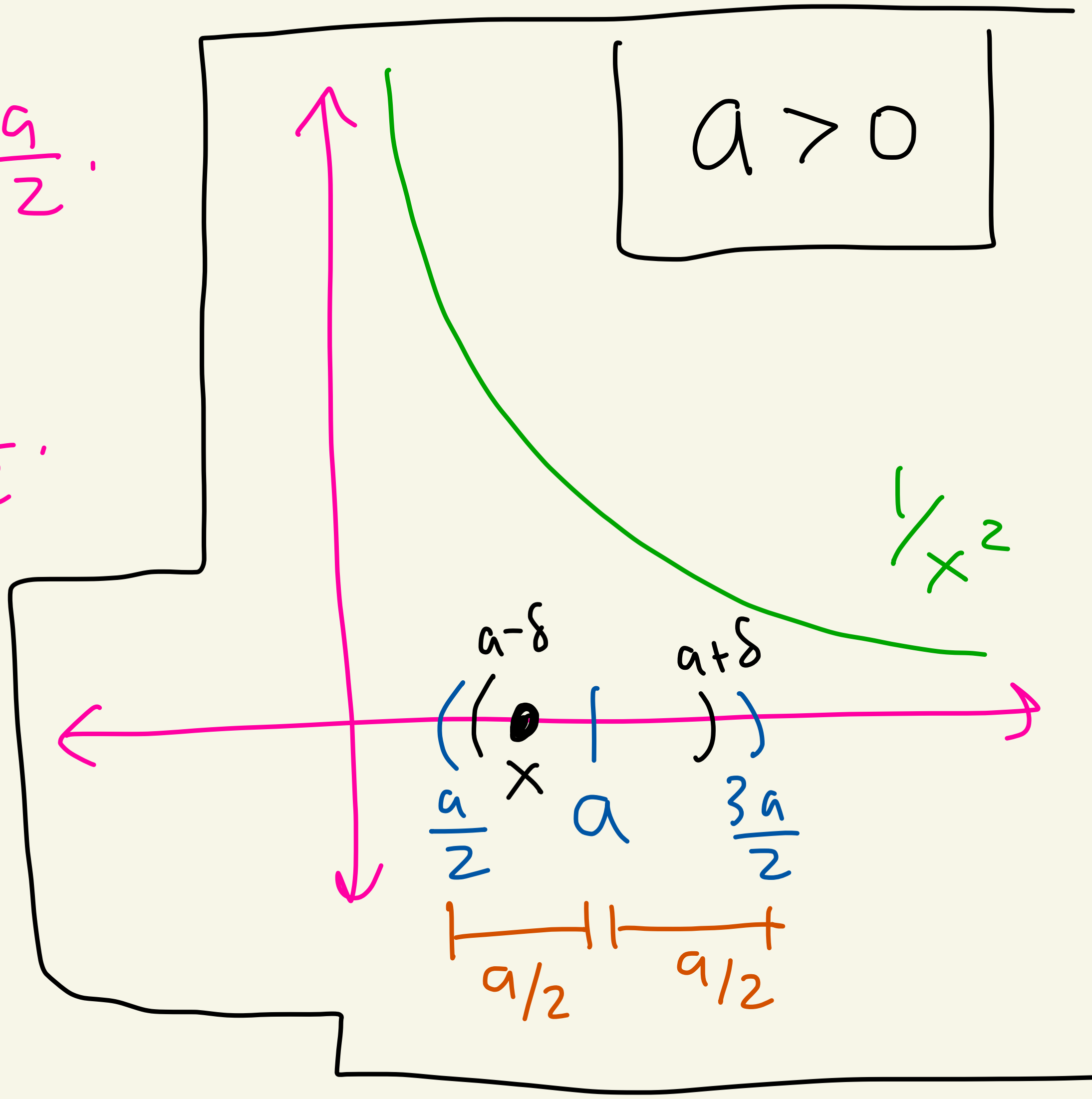
Note that

$$\left| \frac{1}{x^2} - \frac{1}{a^2} \right| = \left| \frac{a^2 - x^2}{x^2 a^2} \right|$$

$$= \frac{|x^2 - a^2|}{|x^2||a|^2}$$

$$= \frac{|x-a||x+a|}{|x|^2 |a|^2}$$

$$= |x-a| \cdot \frac{1}{|x|^2} \cdot \frac{1}{|a|^2} \cdot |x+a|$$



Assume $\delta \le \frac{a}{2}$.

Suppose

$|x-a| < \delta \le \frac{a}{2}$.

Then

$-\frac{a}{2} < x-a < \frac{a}{2}$.

Then

$$\boxed{\frac{a}{2} < x < \frac{3a}{2}}$$

So,

$$\frac{2}{a} > \frac{1}{x} > \frac{2}{3a}$$

And

$$\frac{4}{a^2} > \frac{1}{x^2} > \frac{4}{9a^2} \longleftarrow \boxed{\frac{1}{|x|^2} < \frac{4}{a^2}}$$

Also,

$$a + \frac{a}{2} < x + a < \frac{3a}{2} + a$$

$$\frac{3a}{2} < x + a < \frac{5a}{2} \longleftarrow \boxed{|x+a| < \frac{5a}{2}}$$

Then, if $|x-a| < \delta \leq \frac{a}{2}$, then

$$\left|\frac{1}{x^2} - \frac{1}{a^2}\right| = |x-a| \cdot \frac{1}{|x|^2} \cdot \frac{1}{|a|^2} \cdot |x+a|$$

$$< |x-a| \cdot \frac{4}{a^2} \cdot \frac{1}{a^2} \cdot \frac{5a}{2}$$

$$= |x-a| \cdot \frac{10}{a^3}$$

Set $\delta = \min\left\{\frac{a}{2}, \varepsilon/(10/a^3)\right\}$.

Then, $\delta \leq \frac{a}{2}$ and $\delta \leq \varepsilon/(10/a^3)$.

If $|x-a| < \delta$, then

$$\left|\frac{1}{x^2} - \frac{1}{a^2}\right| < |x-a| \cdot \frac{10}{a^3}$$

$$\boxed{\delta \le \frac{a}{2}} \quad < \frac{\varepsilon}{\left(\frac{10}{a^3}\right)} \cdot \frac{10}{a^3} = \varepsilon$$

$$\boxed{\delta \le \varepsilon / (10/a^3)}$$

$\blacksquare$

$\boxed{8(b)}$

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n} \leftarrow \boxed{\text{alternating harmonic series}}$$

converges by alt. series test

But

$$\sum_{n=1}^{\infty} \left| (-1)^n \cdot \frac{1}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

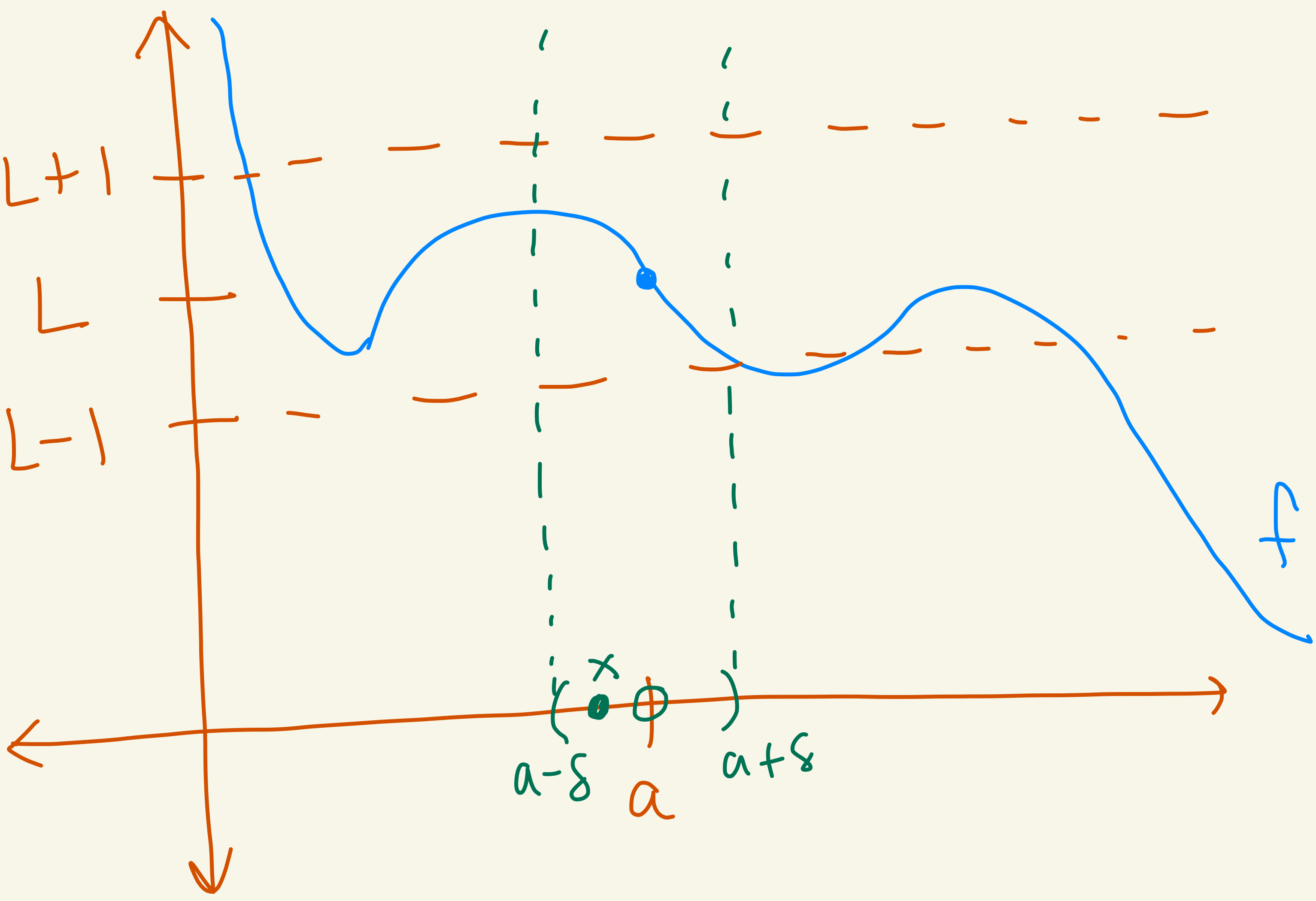
diverges. It's the harmonic series.

HW 4

5) $f: D \to \mathbb{R}$, a limit point of D.

$\lim\limits_{x \to a} f(x)$ exists

Find $M > 0$, $\delta > 0$ where $|f(x)| < M$ for $0 < |x-a| < \delta$.



Suppose $\lim_{x \to a} f(x) = L$.

Let $\varepsilon = 1$.

Then there exists $\delta > 0$ where

if $0 < |x-a| < \delta$,

then $|f(x) - L| < 1$.

So if $0 < |x-a| < \delta$, then

$$|f(x)| = |f(x) - L + L|$$

$$\leq |f(x) - L| + |L|$$

$$< \underbrace{1 + |L|}_{M = 1 + |L|}$$

$\blacksquare$

Hw 4

7(e)

$$\sum_{n=1}^{\infty}\left(\sqrt{n+1}-\sqrt{n}\right)$$

Let S_k be the k-th partial sum

$$S_1=(\sqrt{2}-\sqrt{1})=\sqrt{2}-1$$

$$S_2=(\cancel{\sqrt{2}}-\sqrt{1})+(\sqrt{3}-\cancel{\sqrt{2}})=\sqrt{3}-1$$

$$S_3=(\cancel{\sqrt{2}}-\sqrt{1})+(\cancel{\sqrt{3}}-\cancel{\sqrt{2}})+(\sqrt{4}-\cancel{\sqrt{3}})=\sqrt{4}-1$$

$\vdots \qquad \vdots$

In general,

$$S_k=\sqrt{k+1}-1$$

But $\lim_{k\to\infty} S_k$ does not exist because it's an unbounded sequence

$$\left(\lim_{k\to\infty} s_k = \infty\right)$$

Let $M > 0$.

Note that

$s_k > M$

iff

$\sqrt{k+1} - 1 > M$

iff

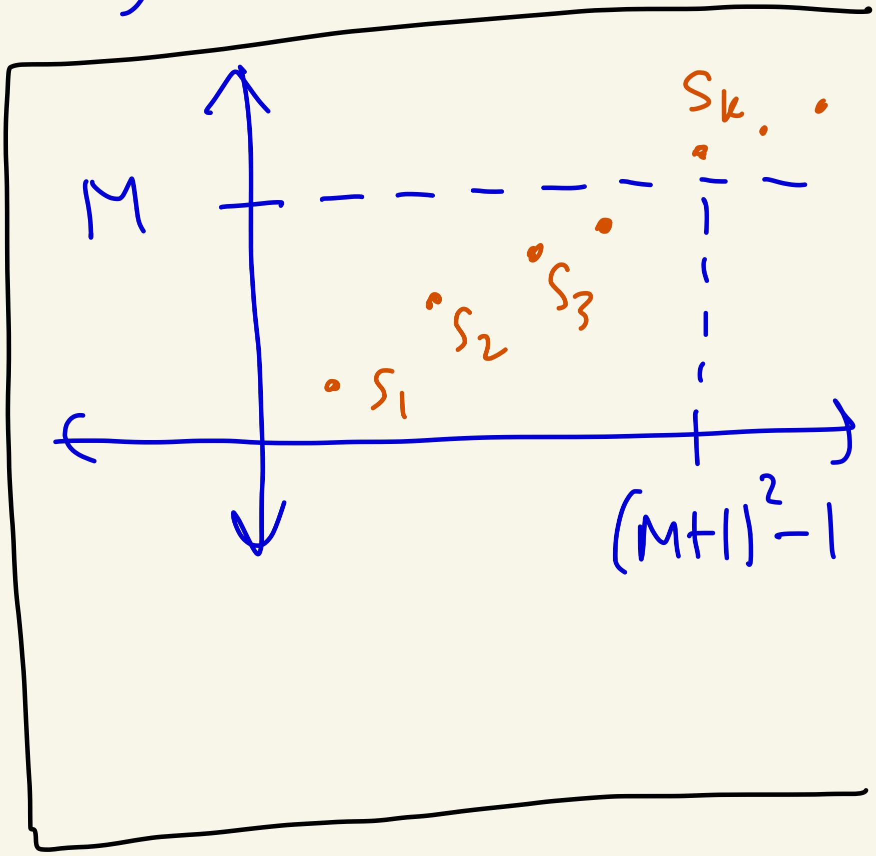
$\sqrt{k+1} > M+1$

iff

$k+1 > (M+1)^2$

iff

$k > (M+1)^2 - 1.$



Given M, if $k > (M+1)^2 - 1$

then $s_k > M$.

So (s_k) is unbounded.