Math 4650

11/12/25

Hw 3

4) Show that

$$\sum_{n=2}^{\infty} \frac{2}{n^2-1} = \frac{3}{2}$$

Find a formula for the k-th partial sum when solving.

$$\frac{2}{n^2-1} = \boxed{\frac{2}{(n-1)(n+1)} = \frac{A}{n-1} + \frac{B}{n+1}}$$

$$2 = A(n+1) + B(n-1)$$

$n=-1$: $2 = A(-1+1) + B(-1-1)$

$$2 = A(0) - 2B$$

$$B = -1$$

$\underline{n=1:}\quad 2 = A(1+1) + B(1-1)$

$$2 = 2A$$

$$A = 1$$

So,

$$\sum_{n=2}^{\infty} \frac{2}{n^2-1} = \sum_{n=2}^{\infty} \left[\frac{1}{n-1} - \frac{1}{n+1} \right]$$

The partial sums are:

$$S_2 = \underbrace{\left[\frac{1}{2-1} - \frac{1}{2+1} \right]}_{n=2} = \frac{1}{1} - \frac{1}{3} = 1 - \frac{1}{3}$$

$$S_3 = \left[\frac{1}{2-1} - \frac{1}{2+1} \right] + \left[\frac{1}{3-1} - \frac{1}{3+1} \right]$$

$$= \left[1 - \frac{1}{3} \right] + \left[\frac{1}{2} - \frac{1}{4} \right]$$

$$= \boxed{1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{4} = \frac{3}{2} - \frac{1}{3} - \frac{1}{4}}$$

$$S_4 = S_3 + \underbrace{\left[\frac{1}{4-1} - \frac{1}{4+1}\right]}_{n=4}$$

$$= \left(1 + \frac{1}{2} - \cancel{\frac{1}{3}} - \frac{1}{4}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{5}\right)$$

$$= \boxed{1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} = \frac{3}{2} - \frac{1}{4} - \frac{1}{5}}$$

$$S_5 = S_4 + \left[\frac{1}{5-1} - \frac{1}{5+1}\right]$$

$$= \left(\frac{3}{2} - \cancel{\frac{1}{4}} - \frac{1}{5}\right) + \left(\cancel{\frac{1}{4}} - \frac{1}{6}\right)$$

$$= \boxed{\frac{3}{2} - \frac{1}{5} - \frac{1}{6}}$$

In general,

$$S_k = \frac{3}{2} - \frac{1}{k} - \frac{1}{k+1}$$

As $k \to \infty$ we get

$$\lim_{k \to \infty} S_k = \frac{3}{2} - 0 - 0 = \frac{3}{2}$$

$\boxed{\text{HW 4}}$

$\boxed{2(c)}$ Show $\lim_{x \to 2} x^4 = 16$

Let $\varepsilon > 0$.

$\boxed{\text{Goal: We want } \delta > 0 \text{ where if } 0 < |x-2| < \delta \text{ then } |x^4 - 16| < \varepsilon}$

Note that

$$|x^4 - 16| = |(x^2-4)(x^2+4)|$$
$$= |(x-2)(x+2)(x^2+4)|$$

$$= \underbrace{|x-2|}_{\text{control with } \delta} \cdot \underbrace{|x+2| \cdot |x^2+4|}_{\text{turn these into \#s}}$$

Assume that $\delta \le 1$.

Suppose $|x-2| < \delta \le 1$.

Then, $\boxed{-1 < x-2 < 1}$

So, $3 < x+2 < 5$.

$+4$

So, $\boxed{|x+2| < 5}$

$+2$

And, $1 < x < 3$

So, $1 < x^2 < 9$

And, $5 < x^2+4 < 13$

Thus, $\boxed{|x^2+4| < 13}$

Therefore if $0 < |x-2| < 1$, then

$$|x^4 - 16| = |x-2||x+2||x^2+4|$$
$$< |x-2| \cdot 5 \cdot 13$$
$$= 65|x-2|$$

Let $\delta = \min\{1, \frac{\varepsilon}{65}\}$.

Then, $\delta \le 1$ and $\delta \le \frac{\varepsilon}{65}$.

If $0 < |x-2| < \delta$, then

$$|x^4 - 16| < 65|x-2| < 65 \cdot \frac{\varepsilon}{65} = \varepsilon$$

$\boxed{\delta \le 1}$ $\boxed{\delta \le \frac{\varepsilon}{65}}$

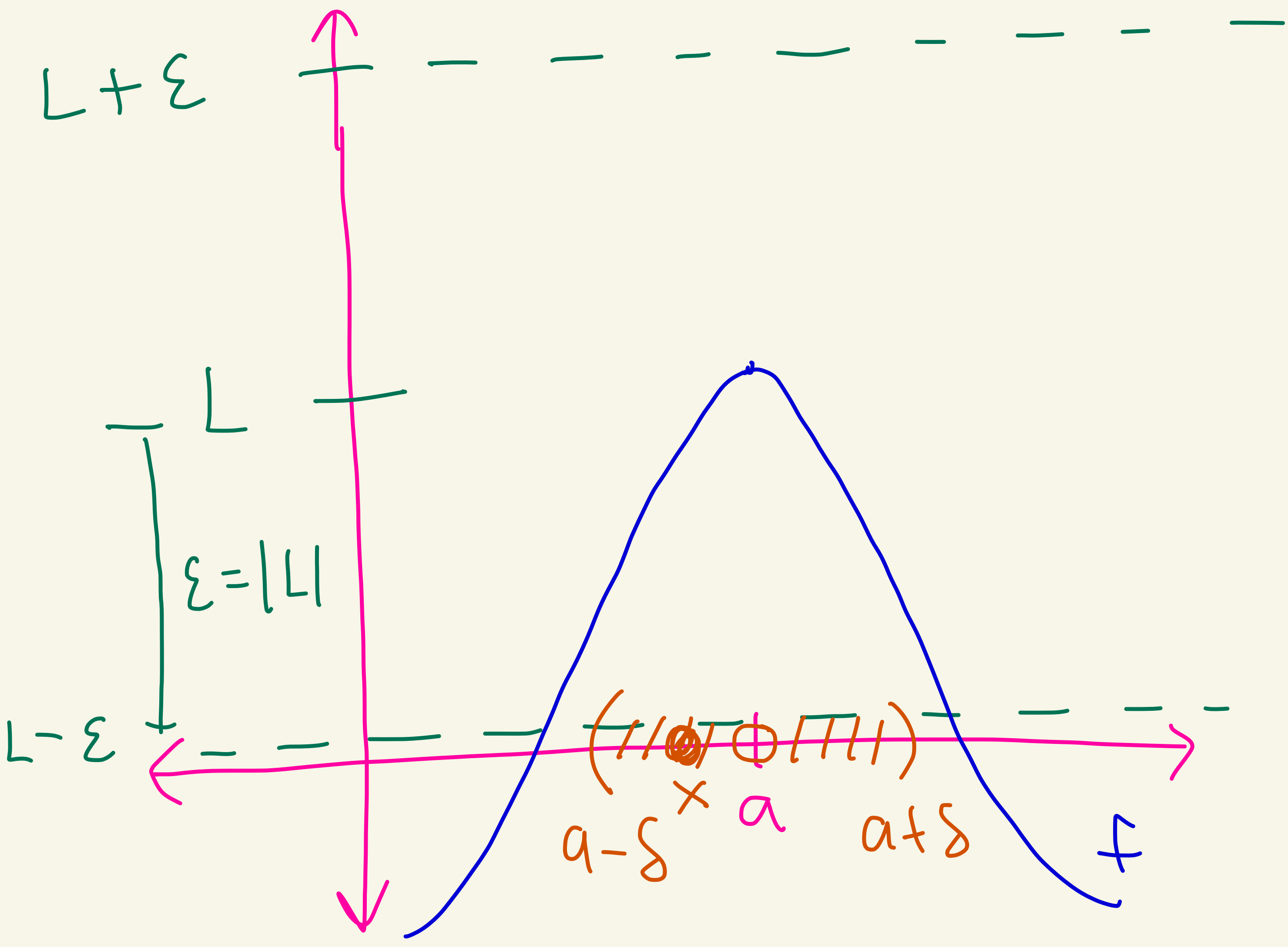
$\blacksquare$

HW 4 #4

Let $f: D \to \mathbb{R}$, a is a limit point of D.

Suppose $\lim_{x \to a} f(x) = L$ where $L \neq 0$.

Prove there exists $\delta > 0$ where if $x \in D$ and $0 < |x-a| < \delta$ then $|f(x)| > 0$



Let $\varepsilon = |L|$.

Since $\lim_{x \to a} f(x) = L$ there exists $\delta > 0$ where if $x \in D$ and $0 < |x-a| < \delta$, then

$$|f(x) - L| < \underbrace{|L|}_{\varepsilon = |L|}$$

So if $x \in D$ and $0 < |x-a| < \delta$, then

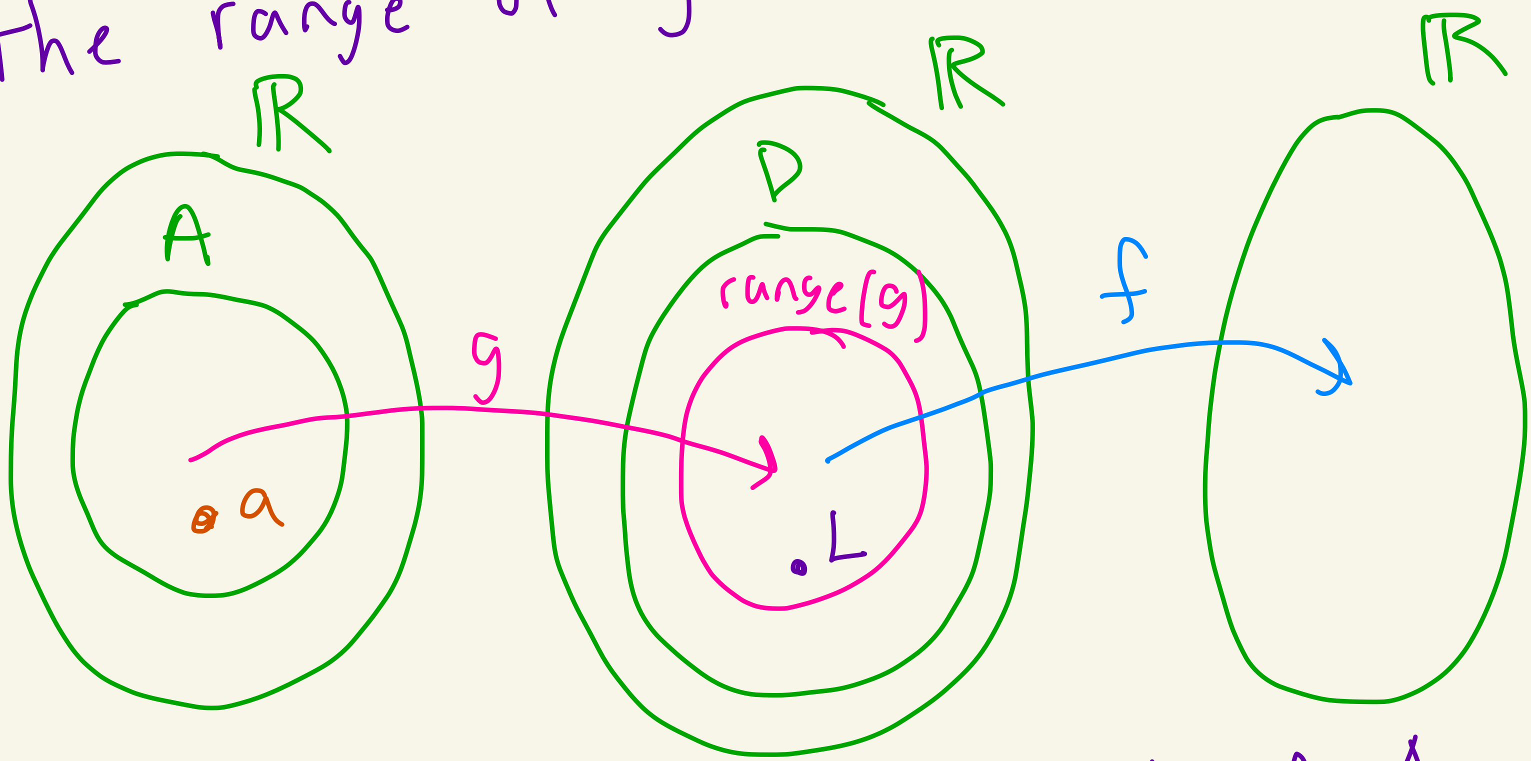
$$\begin{aligned} |L| &= |L - f(x) + f(x)| \\ &\leq |L - f(x)| + |f(x)| \\ &< |L| + |f(x)| \end{aligned}$$

$|L| < |L| + |f(x)|$

Thus, if $x \in D$ and $0 < |x-a| < \delta$, then $0 < |f(x)|$ $\blacksquare$

$\boxed{\text{HW 5}}$ 5

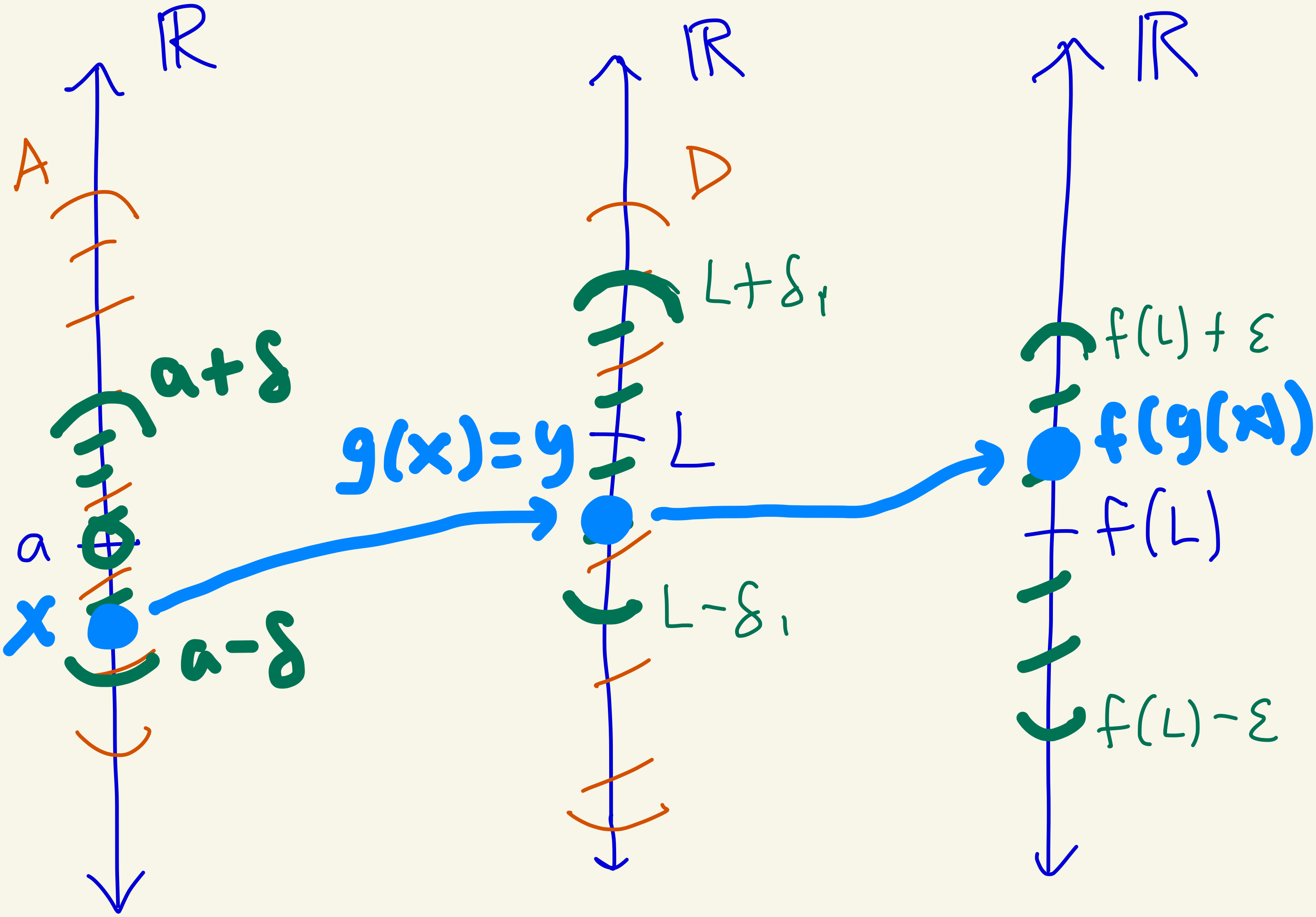
Suppose that $f: D \to \mathbb{R}$ is continuous on D. Suppose $g: A \to \mathbb{R}$ where the range of g is contained in D.



Suppose a is a limit point of A, and $\lim_{x\to a} g(x) = L$ with $L \in D$.

Prove:

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(L)$$



Let $\varepsilon > 0$,

Since f is continuous at L, there exists $\delta_1 > 0$ where if $|y-L| < \delta_1$ then $|f(y)-f(L)| < \varepsilon$

Since $\lim_{x \to a} g(x) = L$ there exists $\delta > 0$ where if $x \in A$ and $0 < |x-a| < \delta$, then $|g(x)-L| < \delta_1$.

So if $x \in A$ and $0 < |x-a| < \delta$, then $|f(g(x))-f(L)| < \varepsilon$

Thus, $\lim_{x \to a} f(g(x)) = f(L)$ $\blacksquare$