Math 4650

11/10/25

$\boxed{\text{HW 3}}$

1

$$\sum_{n=2}^{\infty}\left(\frac{2}{7}\right)^n = \left(\frac{2}{7}\right)^2+\left(\frac{2}{7}\right)^3+\left(\frac{2}{7}\right)^4+\cdots$$

$$=\left(\frac{2}{7}\right)^2\left[1+\left(\frac{2}{7}\right)+\left(\frac{2}{7}\right)^2+\cdots\right]$$

$$=\left(\frac{2}{7}\right)^2\left[\frac{1}{1-\frac{2}{7}}\right]$$

$$\boxed{\begin{array}{l} 1+r+r^2+r^3+\cdots=\dfrac{1}{1-r} \\ -1<r<1 \\ r=2/7 \end{array}}$$

$$=\frac{2^2}{7^2}\left[\frac{7}{5}\right]$$

$$=\boxed{\frac{4}{35}}$$

$\boxed{\text{HW 3 \# 7(d)}}$

Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{2n^3+1}$$

HW 3

\#6 $\lim_{n\to\infty} a_n = 0$ iff $\lim_{n\to\infty} |a_n| = 0$

Note

$$\lim_{n\to\infty} \left| (-1)^n \frac{n^3}{2n^3+1} \right| = \lim_{n\to\infty} \overbrace{\left|(-1)^n\right|}^{1} \left| \frac{n^3}{2n^3+1} \right|$$

$$= \lim_{n\to\infty} \frac{n^3}{2n^3+1} = \lim_{n\to\infty} \frac{1}{2 + 1/n^3}$$

$$= \frac{1}{2+0} = \frac{1}{2} \neq 0$$

So,

$$\lim_{n\to\infty} (-1)^n \frac{n^3}{2n^3+1} \neq 0$$

Thus, $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{2n^3+1}$ diverges

by the divergence test

$\boxed{\text{HW 4 - 2(b)}}$

Prove: $\displaystyle\lim_{x\to 1} \frac{5x}{x+3} = \frac{5}{4}$

Use $\varepsilon-\delta$ def.

<u>proof:</u>

Let $\varepsilon > 0$.

Then,

$$\left| \frac{5x}{x+3} - \frac{5}{4} \right|$$

$$= \left| \frac{20x - 5(x+3)}{4(x+3)} \right|$$

$$= \left| \frac{15x - 15}{4(x+3)} \right| = \left| \frac{15}{4} \right| \left| \frac{x-1}{x+3} \right|$$

$$= \frac{15}{4} \cdot |x-1| \cdot \frac{1}{|x+3|}$$

Suppose $\delta \leq 1$.

Assume $|x-1| < \delta \leq 1$.

$|x| < c$

$-c < x < c$

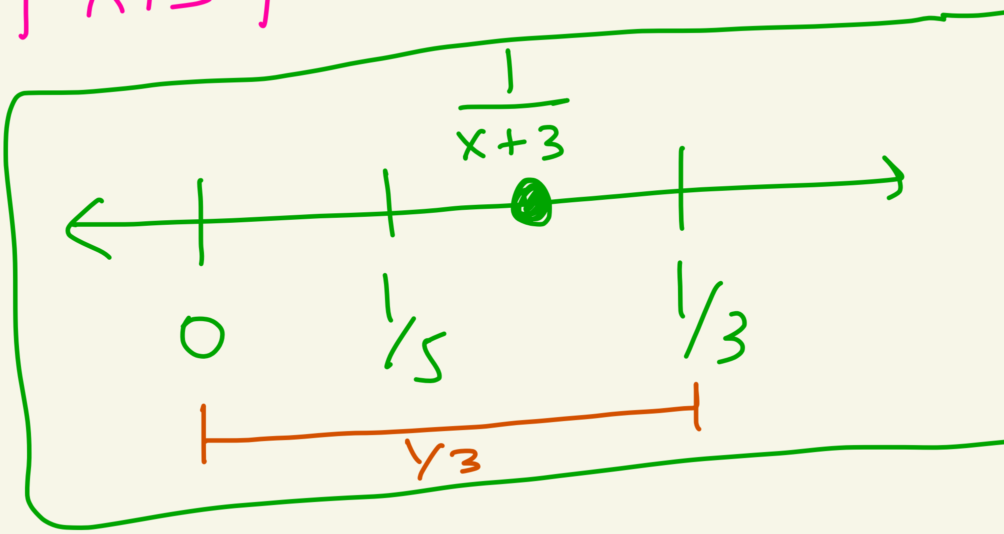
Then, $-1 < x-1 < 1$

$+4$

Then, $3 < x+3 < 5$.

So, $\frac{1}{3} > \frac{1}{x+3} > \frac{1}{5}$

Then, $\frac{1}{|x+3|} = \left|\frac{1}{x+3}\right| < 1/3$



So, if $|x-1| < \delta \leq 1$, then

$$\begin{aligned} \left|\frac{5x}{x+3} - \frac{5}{4}\right| &= \frac{15}{4}|x-1| \frac{1}{|x+3|} \\ &< \frac{15}{4}|x-1| \cdot \frac{1}{3} \\ &= \frac{15}{12}|x-1| \end{aligned}$$

Set $\delta = \min\left\{\frac{\varepsilon}{(15/12)}, 1\right\}$.

Then, $\delta \leq \frac{\varepsilon}{(15/12)}$ and $\delta \leq 1$.

So, if $|x-1|<\delta$, then

$$\left|\frac{5x}{x+3}-\frac{5}{4}\right| < \frac{15}{12}|x-1| < \frac{15}{12}\left(\frac{\varepsilon}{15/12}\right)$$

$$= \varepsilon$$

$\boxed{\delta \le 1}$

$\delta \le \varepsilon/(15/12)$

So, if $|x-1|<\delta$, then

$$\left|\frac{5x}{x+3}-\frac{5}{4}\right| < \varepsilon$$

$\boxed{\text{HW 5}}$

1 (c) Show that $f(x)=x^2+x$ is continuous at any $a \in \mathbb{R}$.

[We will show $\lim\limits_{x\to a}(x^2+x)=a^2+a$]

proof:

Let $\varepsilon > 0$.

Note that

$$|(x^2+x)-(a^2+a)|$$

$$= |x^2-a^2+x-a|$$

$$\leq |x^2-a^2|+|x-a|$$

$$= |x-a||x+a|+|x-a|$$

$$= |x-a|(|x+a|+1)$$

Suppose $\delta \leq 1$.

Assume $|x-a| < \delta \leq 1$.

Then,

$$\begin{aligned} |x+a|+1 &= |x-a+a+a|+1 \\ &= |x-a+2a|+1 \\ &\leq |x-a|+|2a|+1 \\ &< 1+2|a|+1 \\ &= 2+2|a| \end{aligned}$$

So if $|x-a| < \delta \leq 1$, then

$$\begin{aligned} &|(x^2+x)-(a^2+a)| \\ &= |x-a|(|x+a|+1) \\ &< |x-a|(2+2|a|) \end{aligned}$$

Let $\delta = \min\left\{\frac{\varepsilon}{2+2|a|}, 1\right\}$

So, $\delta \leq \frac{\varepsilon}{2+2|a|}$ and $\delta \leq 1$.

Thus, if $|x-a|<\delta$, we get

$$\begin{aligned}
|(x^2+x)-(a^2+a)| &< \underbrace{|x-a|}_{\delta \le 1}(2+2|a|) \\
&< \frac{\varepsilon}{(2+2|a|)}(2+2|a|) \quad \boxed{\delta \le \frac{\varepsilon}{2+2|a|}} \\
&= \varepsilon .
\end{aligned}$$

Thus, if $|x-a|<\delta$, we have

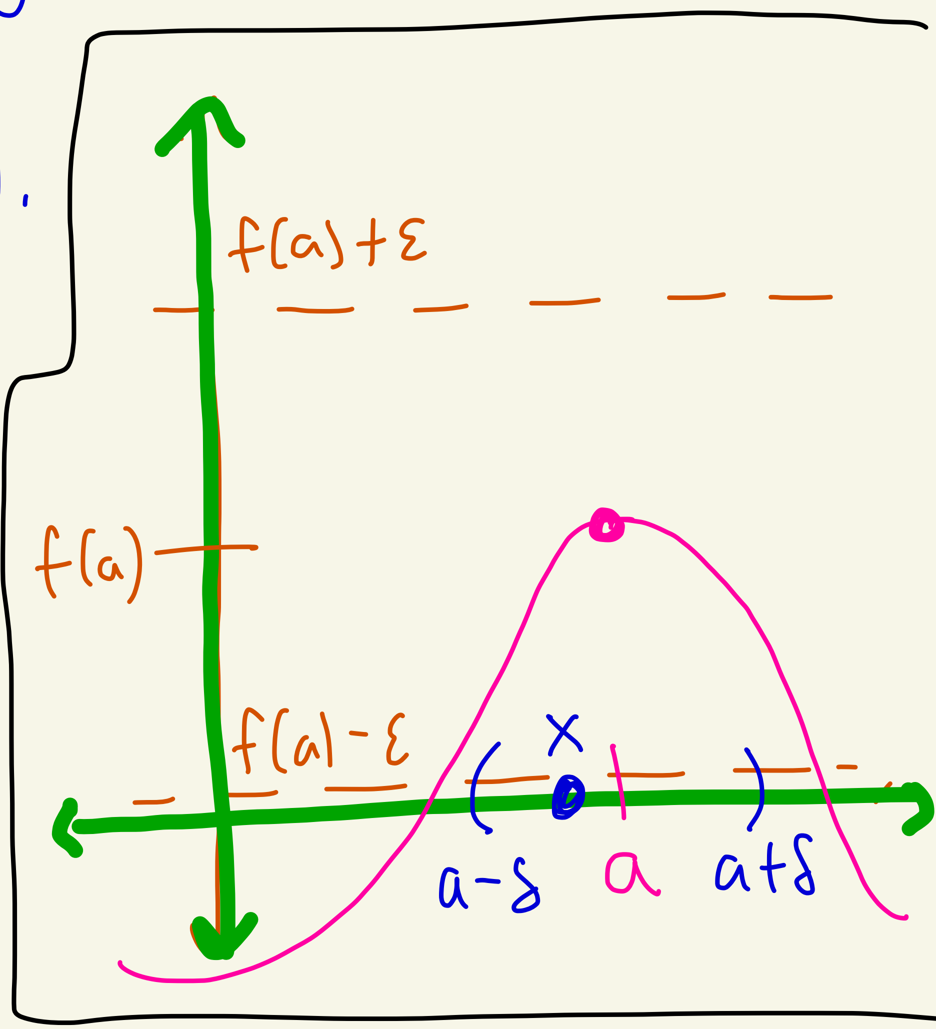
$$|(x^2+x)-(a^2+a)|<\varepsilon$$

$\blacksquare$

HW 5 #3

proof:

We are given that $f:\mathbb{R}\to\mathbb{R}$ is continuous at $a\in\mathbb{R}$ and $f(a)>0$



Let $\varepsilon = f(a)$.

By the def of continuity there exists $\delta>0$ where if $|x-a|<\delta$ then

$$|f(x)-f(a)| < \underbrace{f(a)}_{\varepsilon}$$

So if $|x-a|<\delta$, then

$$-f(a) < f(x) - f(a) < f(a)$$

So if $|x-a| < \delta$, then

$$0 < f(x) < 2f(a).$$

So if $|x-a| < \delta$, then $f(x) > 0$

$\blacksquare$