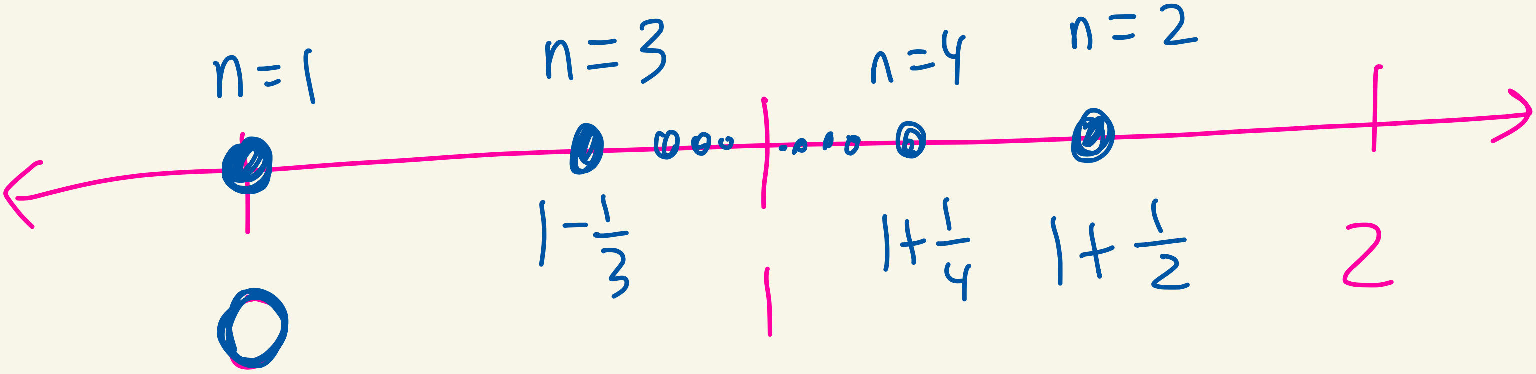
Math 4650

10/8/25

$\boxed{\text{HW 1}}$

2 (b) Find inf/sup.

$$X = \left\{ 1 + \frac{(-1)^n}{n} \;\middle|\; n \in \mathbb{N} \right\}$$

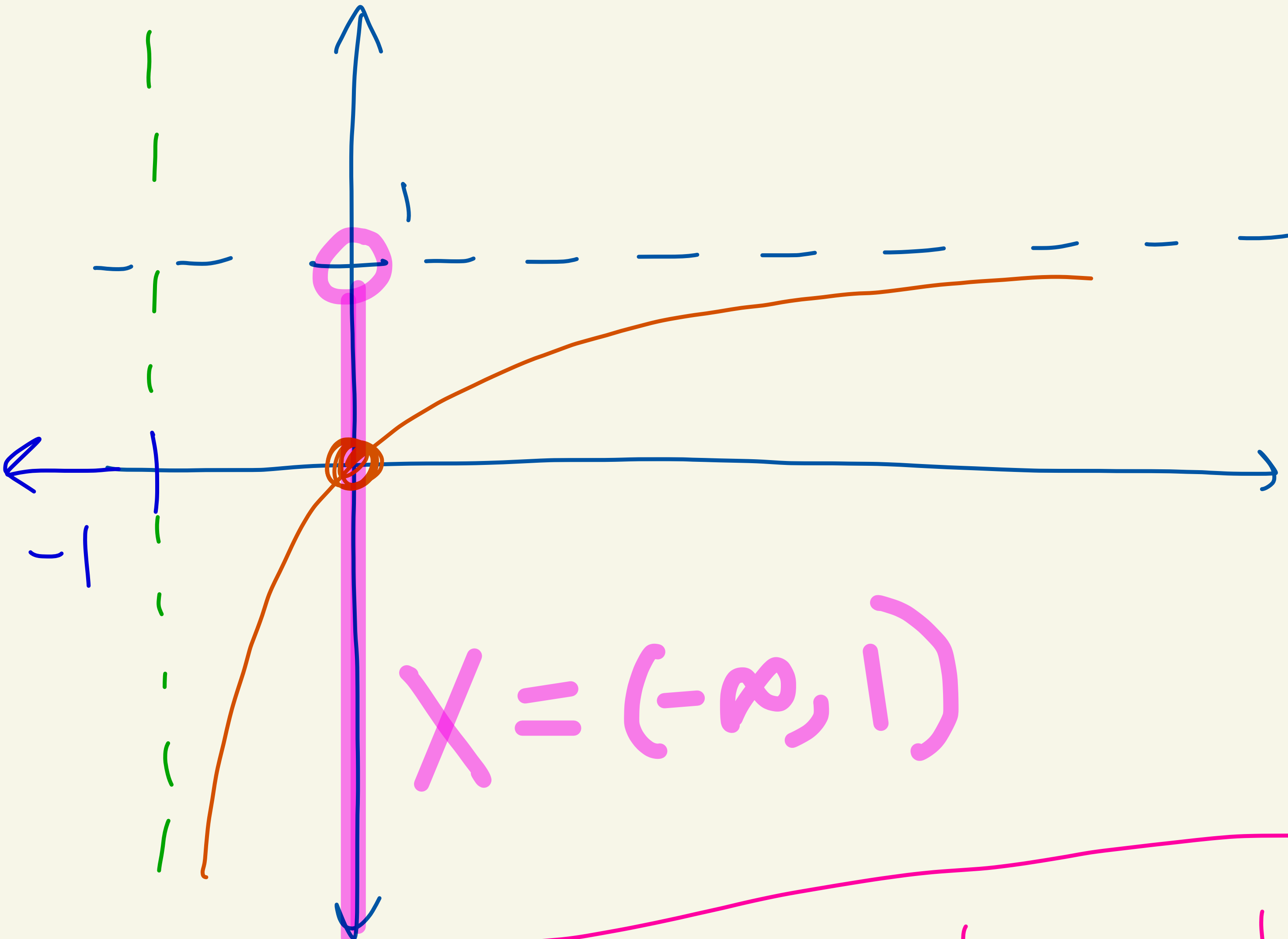


$\inf(X) = 0$

$\sup(X) = 1 + 1/2 = 3/2$

② (d)

$$X = \left\{ \frac{x}{1+x} \;\middle|\; x \in \mathbb{R},\ x > -1 \right\}$$



$$\lim_{x\to\infty} \frac{x}{1+x} = \lim_{x\to\infty} \frac{1}{\frac{1}{x}+1} = \frac{1}{0+1}$$

$$= 1$$

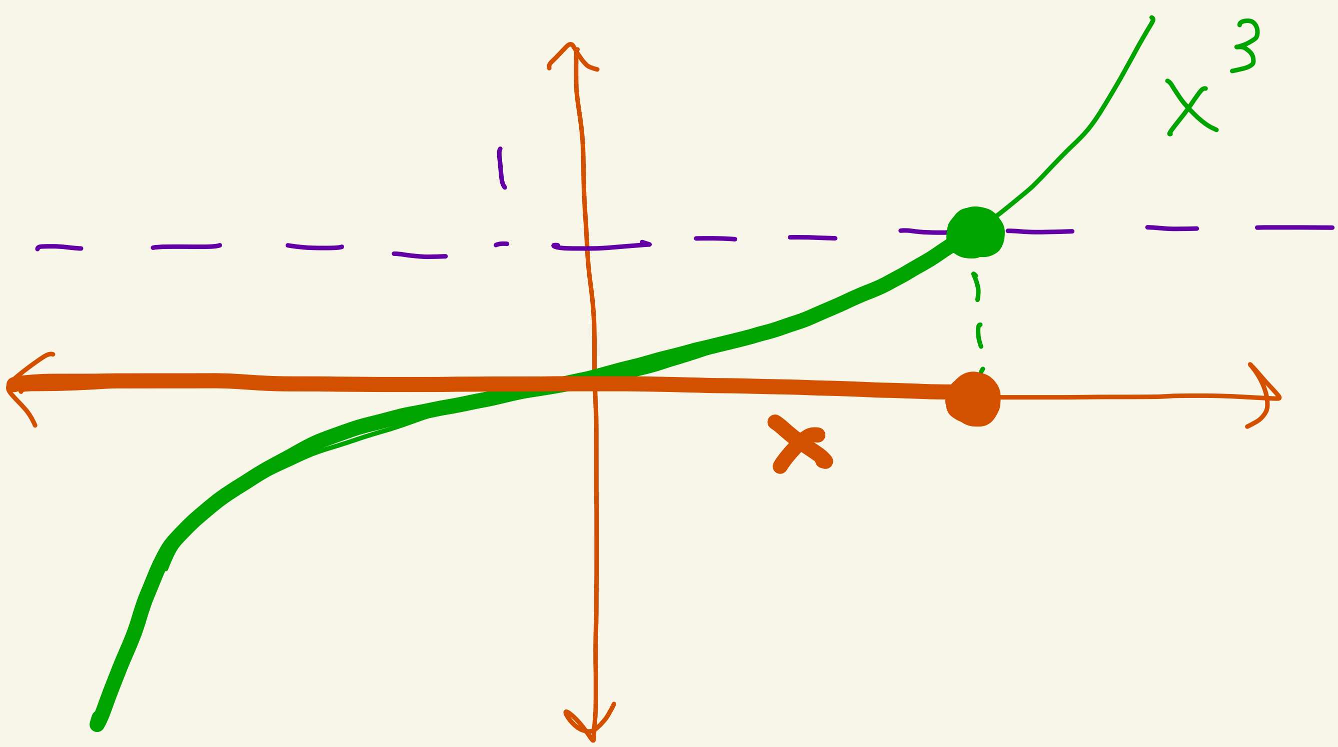
X has no infimum

$\sup(X) = 1$

HW 1

② (f)

$X = \{x \in \mathbb{R} \mid x^3 \leq 1\}$



$X = (-\infty, 1]$

$\inf(X)$ DNE

$\sup(X) = 1$

$\boxed{\text{Hw 1}}$

5 Let $S \subseteq \mathbb{R}$, $S \neq \phi$.

Suppose b is an upper bound for S and $b \in S$.

Prove $b = \sup(S)$.

<u>proof:</u>

Since b is an upper bound for S, by the completeness axium, $\sup(S)$ exists.

We are given that b is an upper bound for S.

Let's show b is the least upper bound for S.

Suppose c is some upper bound for S.

Then, $x \leq c$ for all $x \in S$.

Since $b \in S$ is given we know $b \leq c$.

So, b is the least upper bound for S.

Thus, $b = \sup(S)$ ▨

HW 1

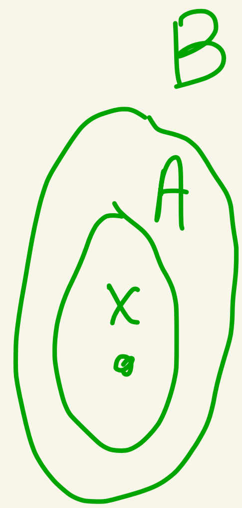
⑥ (a) If $A, B \neq \emptyset$ and both are bounded from below, and $A \subseteq B$, then $\inf(B) \leq \inf(A)$.

proof:

Let $s_B = \inf(B)$, $s_A = \inf(A)$.

Since $s_B = \inf(B)$ we know that

$s_B \leq x$ for all $x \in B$.



Since $A \subseteq B$ we know

$s_B \leq x$ for all $x \in A$.

So, s_B is a lower bound for A.

Since $s_A = \inf(A)$ we know

s_A is the greatest lower bound for A.

So, $s_B \leq s_A$.

Thus, $\inf(B) \leq \inf(A)$ $\blacksquare$

$\boxed{\text{HW 1}}$

7 (a) $A, B \subseteq \mathbb{R}$, $A, B \neq \emptyset$.

$\sup(A), \sup(B)$ exists.

<u>prove:</u>

If $A \cap B \neq \emptyset$, then

$$\sup(A \cap B) \leq \min\{\sup(A), \sup(B)\}$$

<u>proof:</u>

Let $s_A = \sup(A)$, $s_B = \sup(B)$.

Then,

$$x \leq s_A \text{ for all } x \in A$$

and

$$x \leq s_B \text{ for all } x \in B.$$

Let $x \in A \cap B$.

Then, $x \in A$.

So, $x \leq s_A$.

So, s_A is an upper bound for $A \cap B$.

Thus, $s = \sup(A \cap B)$ exists.

Note also that if $x \in A \cap B$ then $x \in B$ implying $x \leq s_B$.

So, s_B is an upper bound for $A \cap B$.

Note that $s = \sup(A \cap B)$ is the least upper bound for $A \cap B$.

Thus, $s \le s_A$ and $s \le s_B$.

So, $s \le \min\{s_A, s_B\}$ $\blacksquare$

$\boxed{\text{HW 2}}$

3 (c)

Show
$$\lim_{n \to \infty} \left(\sqrt{n+1} - \sqrt{n} \right) = 0$$

proof:

Let $\varepsilon > 0$.

Note that
$$\left| \left(\sqrt{n+1} - \sqrt{n} \right) - 0 \right|$$
$$= \left| \sqrt{n+1} - \sqrt{n} \right|$$
$$= \sqrt{n+1} - \sqrt{n}$$

$$= \frac{(\sqrt{n+1} - \sqrt{n})}{1} \frac{(\sqrt{n+1} + \sqrt{n})}{(\sqrt{n+1} + \sqrt{n})}$$

$$= \frac{(n+1) + \cancel{\sqrt{n+1}\sqrt{n}} - \cancel{\sqrt{n}\sqrt{n+1}} - n}{\sqrt{n+1} + \sqrt{n}}$$

$$= \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

$$< \frac{1}{\sqrt{n} + \sqrt{n}}$$

$$= \frac{1}{2\sqrt{n}}$$

$$< \frac{1}{\sqrt{n}}$$

So, $\left|\sqrt{n+1}-\sqrt{n}\right| < \frac{1}{\sqrt{n}}$

We want $\frac{1}{\sqrt{n}} < \varepsilon$.

We have $\frac{1}{\sqrt{n}} < \varepsilon$ iff $\frac{1}{\varepsilon} < \sqrt{n}$

iff $\frac{1}{\varepsilon^2} < n$.

Pick N where $N > \frac{1}{\varepsilon^2}$.

Then, if $n \geq N > \frac{1}{\varepsilon^2}$ we have

$$\left|\left(\sqrt{n+1}-\sqrt{n}\right) - 0\right| < \frac{1}{\sqrt{n}} < \varepsilon.$$

$\blacksquare$