Math 4650

10/6/25

Topic 4 - Limits of functions

$\boxed{\text{2450/3450 Notation}}$

$f : D \to \mathbb{R}$

means f is a function with domain D and range of f contained in $\mathbb{R}$ (outputs of f are in $\mathbb{R}$)

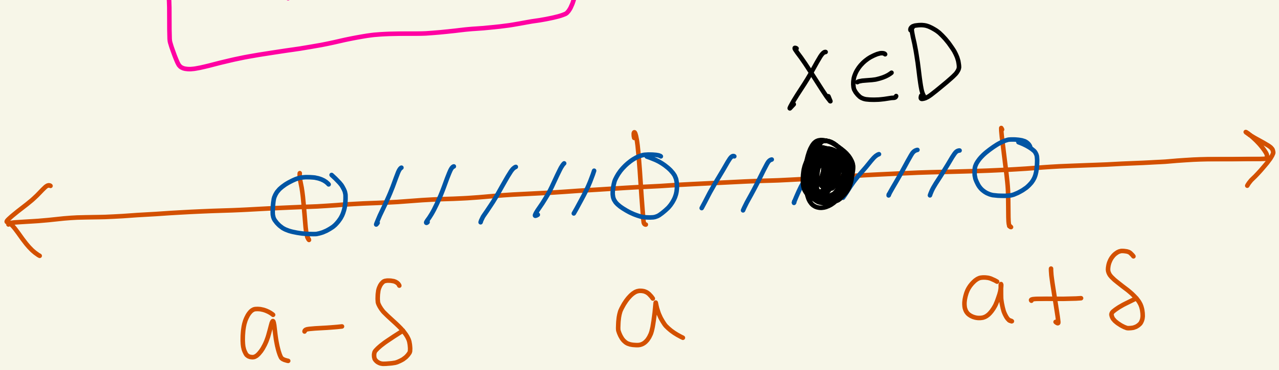
Ex: $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$

$f : \mathbb{R} - \{0\} \to \mathbb{R}$, $f(x) = \frac{1}{x}$

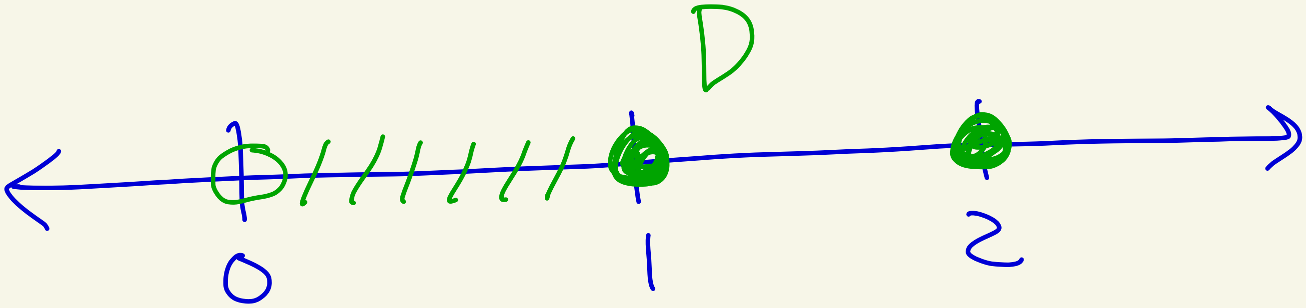
Def: Let $D \subseteq \mathbb{R}$.
Let $a \in \mathbb{R}$.
We say that a is a **limit point** of D if for every $\delta > 0$ there exists $x \in D$ with $0 < |x-a| < \delta$.

$0 < |x-a|$ ensures $x \neq a$

$|x-a| < \delta$
$a - \delta < x < a + \delta$



Ex: $D = (0,1] \cup \{2\}$

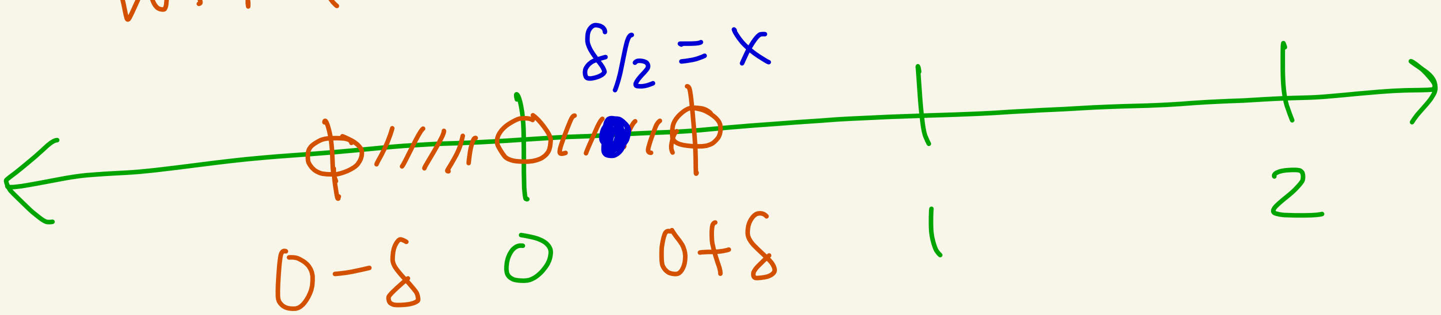


Claim: 0 is a limit point of D

proof:

Let $\delta > 0$.

We must find an $x \in D$ with $0 < |x - 0| < \delta$.



Let $x = \min\left\{\frac{\delta}{2}, \frac{1}{2}\right\}$

Then $x \in D$ and $0 < |x - 0| < \delta$.

So, 0 is a limit point of D. $\blacksquare$

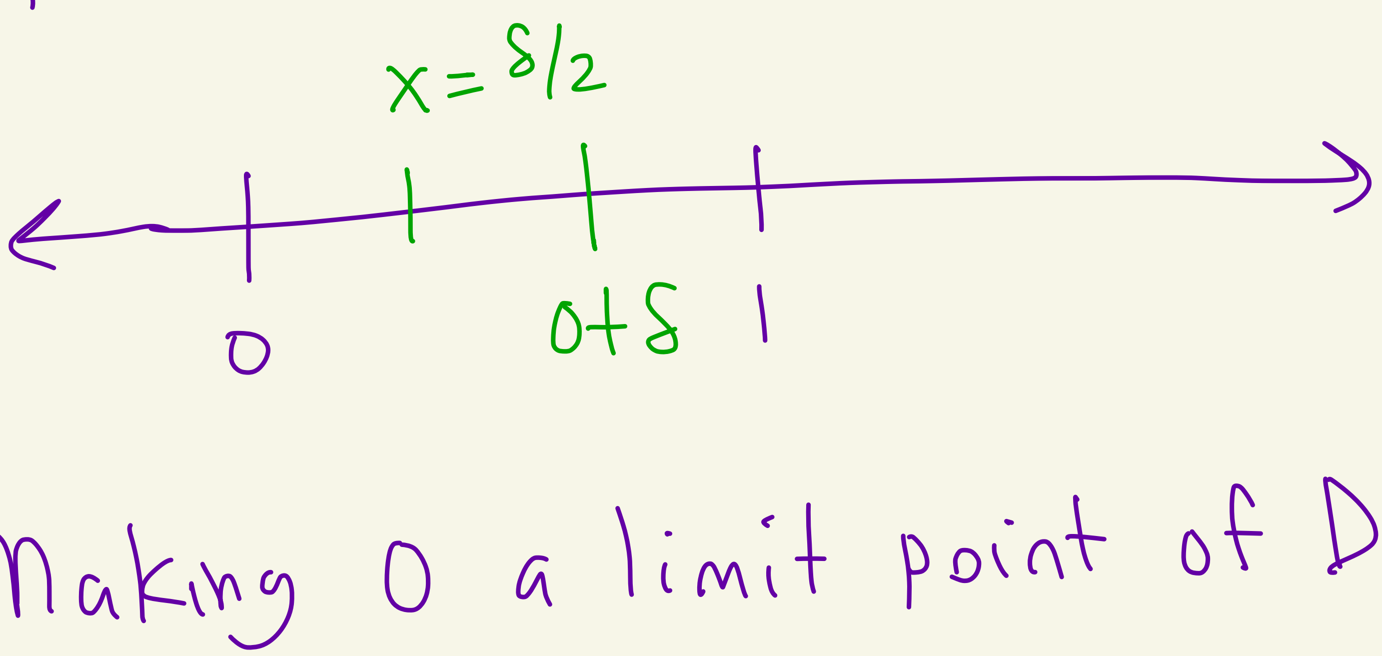
<u>Simplify above argument:</u>

<u>pf:</u> Let $0 < \delta < 1$.

Then set $x = \frac{\delta}{2}$.

So, $0 < x < \frac{1}{2}$

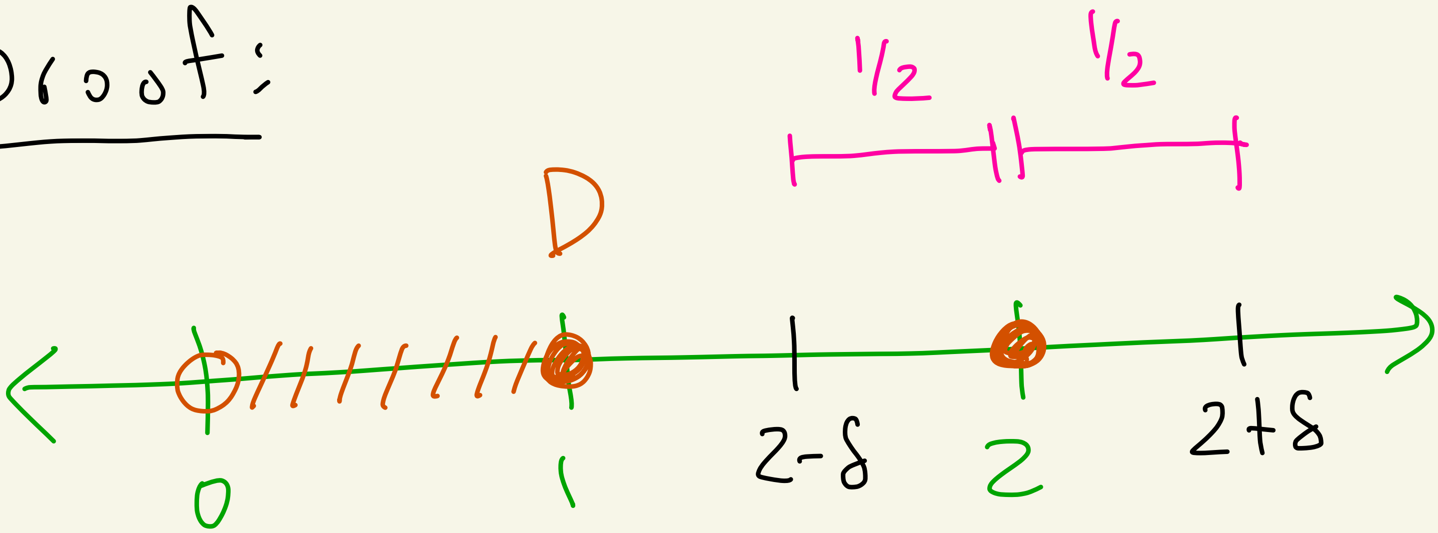
And so $x \in D$ and $0 < |x - 0| < \delta$.



Making 0 a limit point of D.

<u>Claim:</u> 2 is not a limit point of D.

<u>proof:</u>



Set $\delta = \frac{1}{2}$.

There is no $x \in D$ with $0 < |x-2| < \underbrace{\frac{1}{2}}_{\delta}$.

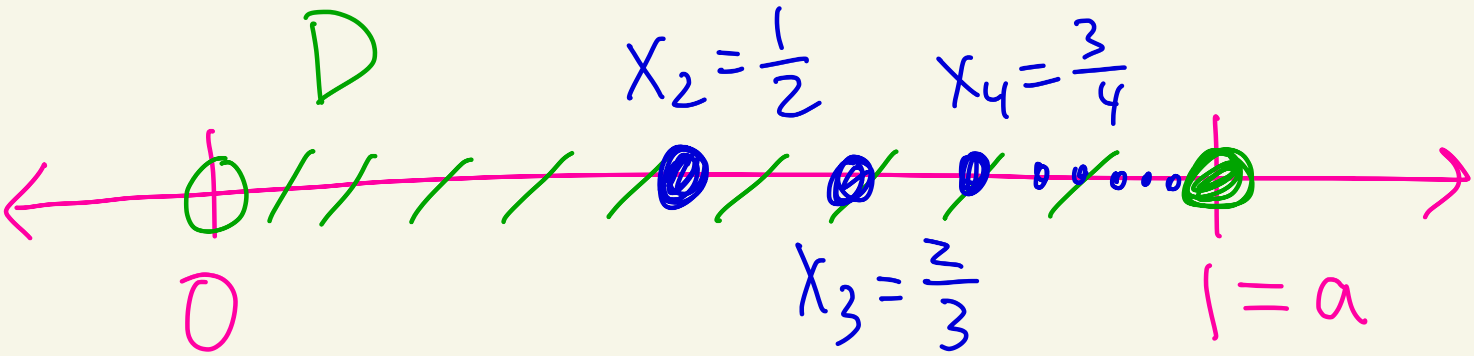
So, 2 is not a limit point of D. ▨

Theorem: Let $D \subseteq \mathbb{R}$ and $a \in \mathbb{R}$. Then: a is a limit point of D iff there exists a sequence $(x_n)_{n=1}^{\infty}$ contained in D, with $x_n \neq a$ for all n, and $\lim_{n\to\infty} x_n = a$.

Proof: HW. $\blacksquare$

Ex: $D = (0,1]$

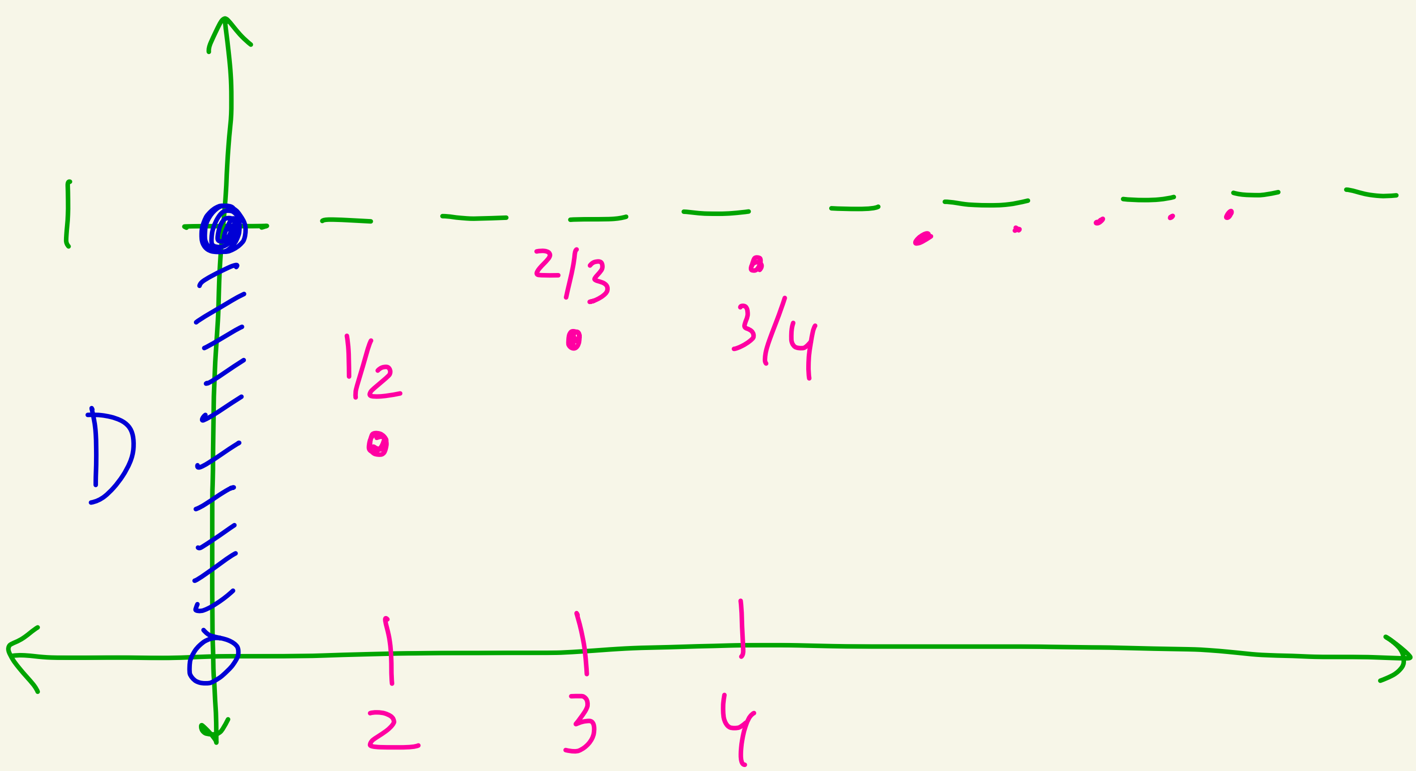
We claim that 1 is a limit point of D.



Let $x_n = 1 - \frac{1}{n}$, $n \geq 2$

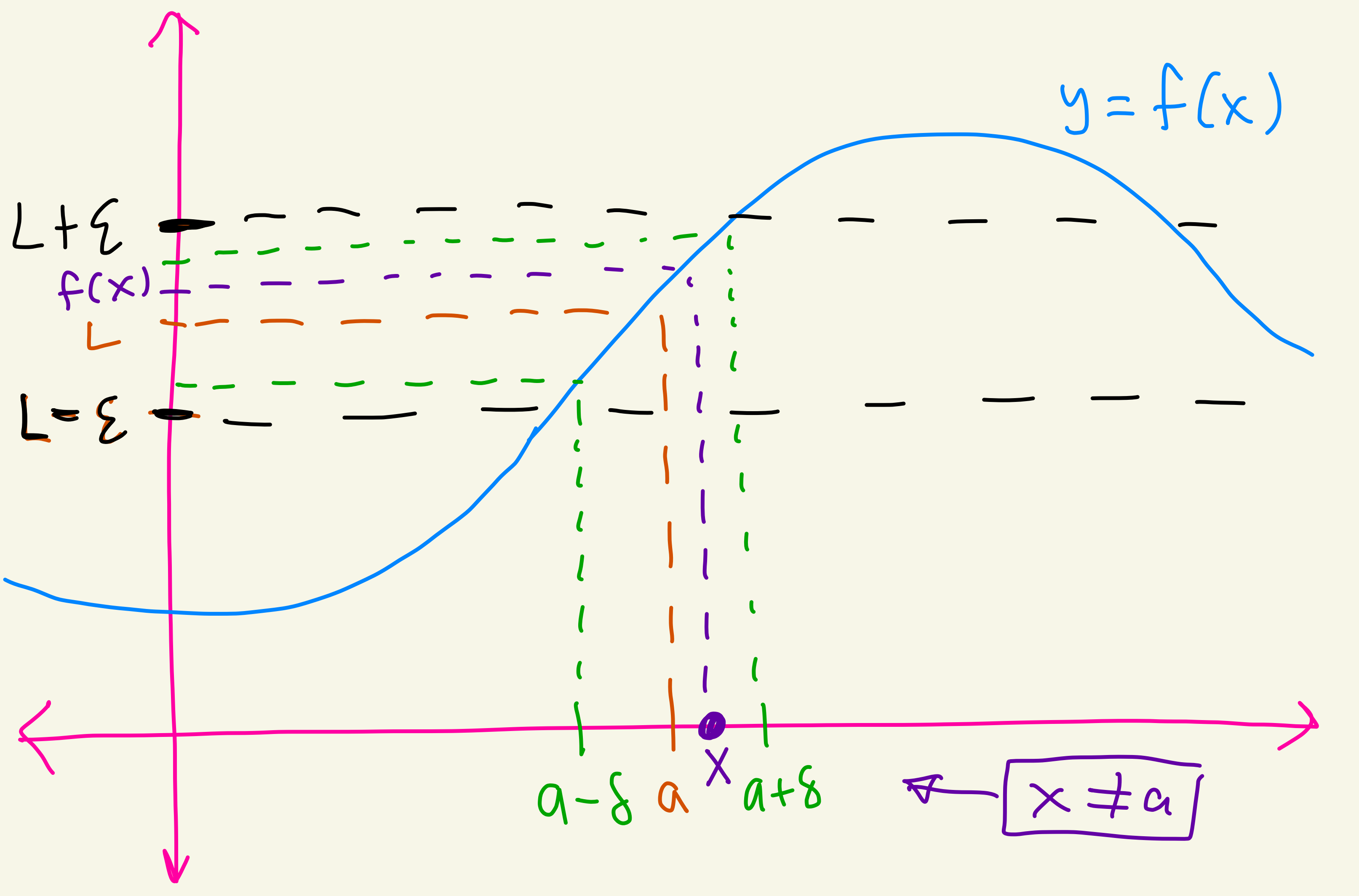
Then $(x_n)_{n=2}^{\infty}$ is a sequence from D, and $x_n \neq 1$ for all $n \geq 2$, and $\lim_{n \to \infty} x_n = 1 - 0 = 1$.

So, 1 is a limit point of D



Def: Let $D \subseteq \mathbb{R}$ and $f: D \to \mathbb{R}$. Let a be a limit point of D. Let $L \in \mathbb{R}$.

We write $\lim\limits_{x \to a} f(x) = L$ if for every $\varepsilon > 0$ there exists $\delta > 0$ where if $x \in D$ and $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.



Ex: Let's show $\lim_{x \to 2} x^2 = 4$

proof:

Let $\varepsilon > 0$.

We want $\delta > 0$ where if $0 < |x-2| < \delta$, then $|x^2-4| < \varepsilon$

Note that

$$|x^2-4| = \underbrace{|x-2|}_{\text{can control with } \delta}\ \underbrace{|x+2|}_{\text{let's work on this term first by restricting } \delta}$$

Suppose $\delta \leq 1$. ← $\boxed{\text{arbitrary choice}}$

Then if $0 < |x-2| < \delta \leq 1$, we

get
$$
\begin{aligned}
|x+2| &= |x-2+4| \\
\boxed{\Delta} \quad &\le |x-2| + |4| \\
&\le 1 + 4 \\
&= 5.
\end{aligned}
$$

So, if $0 < |x-2| < \delta \le 1$, then

$$
\begin{aligned}
|x^2-4| &= |x-2|\,|x+2| \\
&\le |x-2| \cdot 5 \\
&= 5|x-2|.
\end{aligned}
$$

Let $\delta = \min\left\{1, \frac{\varepsilon}{5}\right\}$

So, $\delta \le 1$ and $\delta \le \frac{\varepsilon}{5}$.

Then, if $0 < |x-2| < \delta$,

we get

$$
\begin{aligned}
|x^2-4| &= |x-2|\,|x+2| \\
&\le 5|x-2| \\
&< 5\cdot\delta \\
&\le 5\cdot\frac{\varepsilon}{5} \\
&= \varepsilon
\end{aligned}
$$

$\boxed{\delta \le 1}$

$|x-2| < \delta$

$\delta \le \frac{\varepsilon}{5}$

Hence if $0 < |x-2| < \delta$,

then $|x^2-4| < \varepsilon$

So, $\lim\limits_{x\to 2} x^2 = 4$ $\blacksquare$