Math 4650
10/29/25

Theorem: Let $D \subseteq \mathbb{R}$ and $a \in D$.
Let $f: D \to \mathbb{R}$ and $g: D \to \mathbb{R}$ both be continuous at a. Let $\alpha \in \mathbb{R}$.
Then,

$$\alpha f, \; f+g, \; f-g, \; fg$$

are all continuous at a.
If $g(a) \neq 0$, then $\frac{f}{g}$ is continuous at a.

Proof:
If a is not a limit point of D, then all the above functions are continuous at a.

Suppose a is a limit point of D.
This result will follow from the theorems on limits.

For example, since f and g are continuous at a we get $\lim\limits_{x \to a} f(x) = f(a)$

and $\lim\limits_{x \to a} g(x) = g(a)$.

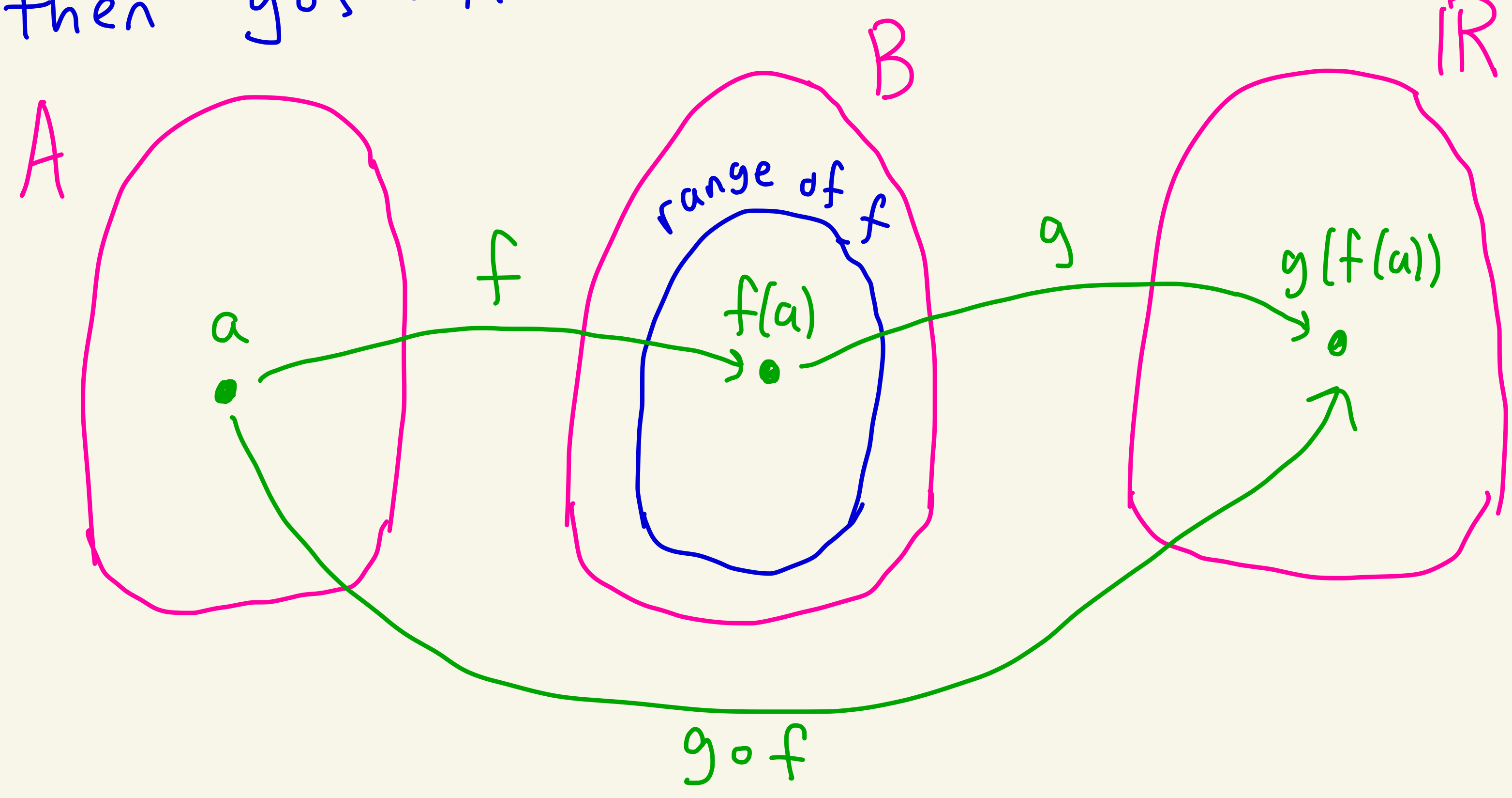
Then by our theorems on limits we get

$$\lim_{x \to a} [f(x)g(x)] = \left[\lim_{x \to a} f(x)\right]\left[\lim_{x \to a} g(x)\right]$$

$$= f(a) g(a)$$

So, fg is continuous at a. $\boxed{}$
The other proofs are similar.

Theorem: Let $A, B \subseteq \mathbb{R}$. Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ where the range of f is contained in B. If f is continuous at some point $a \in A$ and g is continuous at $f(a) \in B$, then $g \circ f : A \to \mathbb{R}$ is continuous at a.



Proof:

Let $\varepsilon > 0$.

Since g is continuous at $f(a)$ there exists $\delta_1 > 0$ where if $y \in B$ and $|y - f(a)| < \delta_1$, then $|g(y) - g(f(a))| < \varepsilon$.

Since f is continuous at a there exists $\delta > 0$ where if $x \in A$ and $|x - a| < \delta$, then $|f(x) - f(a)| < \delta_1$

Since the range of f is contained in B we have that if $x \in A$ and $|x - a| < \delta$, then $f(x) \in B$ and $|f(x) - f(a)| < \delta_1$ which will give $|g(f(x)) - g(f(a))| < \varepsilon$

So if $x \in A$ and $|x - a| < \delta$, then $|g(f(x)) - g(f(a))| < \varepsilon$.

So, $g \circ f$ is continuous at a. $\blacksquare$

Theorem: Let $D \subseteq \mathbb{R}$ and $f: D \to \mathbb{R}$ and $a \in D$. Then, f is continuous at a if and only if $\lim_{n\to\infty} f(x_n) = f(a)$

for every sequence (x_n) contained in D with $\lim_{n\to\infty} x_n = a$.

Proof: HW $\blacksquare$