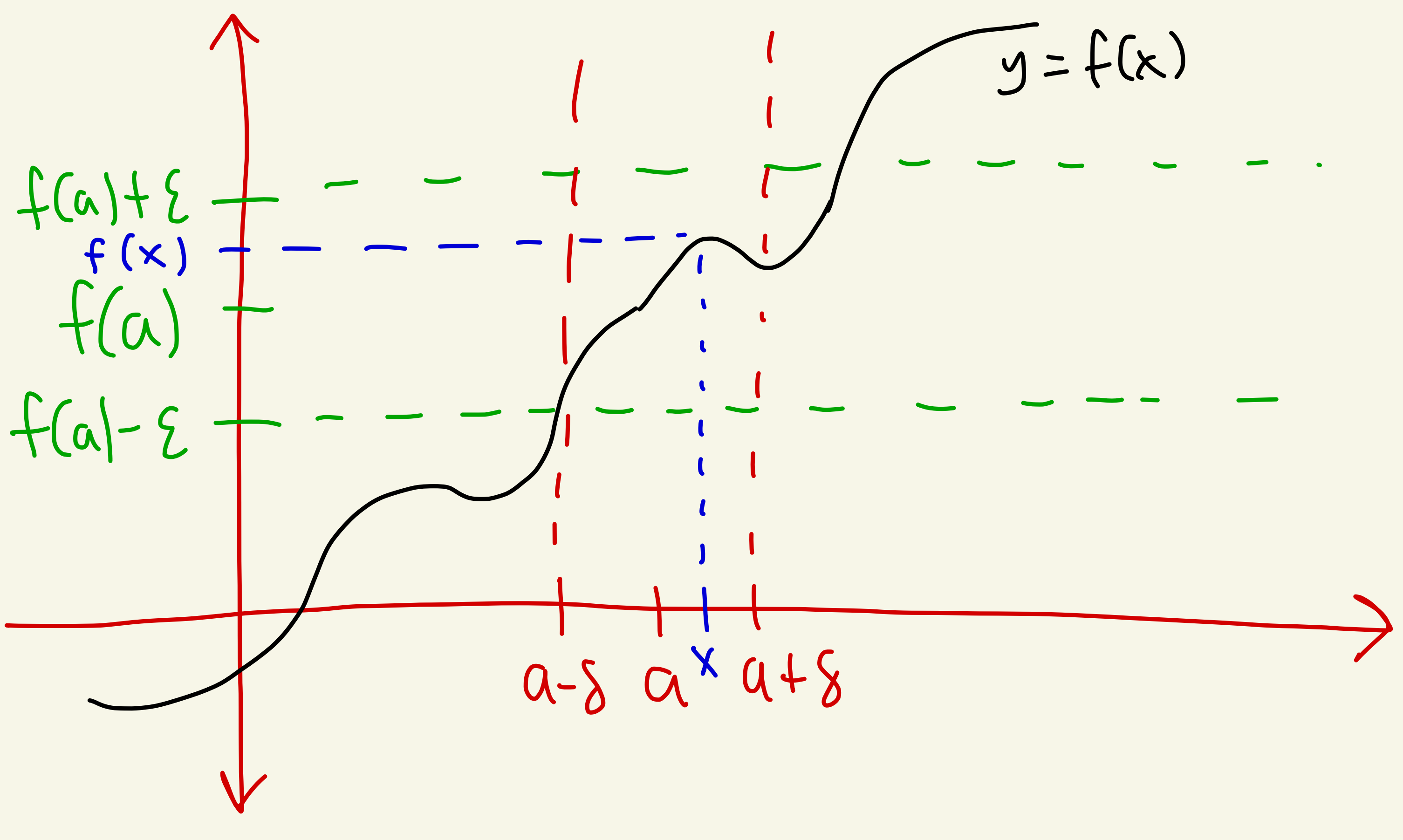
Math 4650

10/27/25

Topic 5 - Continuity

Def: Let $D \subseteq \mathbb{R}$, $a \in D$, $f : D \to \mathbb{R}$. We say that f is continuous at a if for every $\varepsilon > 0$ there exists $\delta > 0$ so that if $x \in D$ and $|x-a| < \delta$, then $|f(x) - f(a)| < \varepsilon$



If $A \subseteq D$, then f is continuous on A if f is continuous at all $a \in A$.

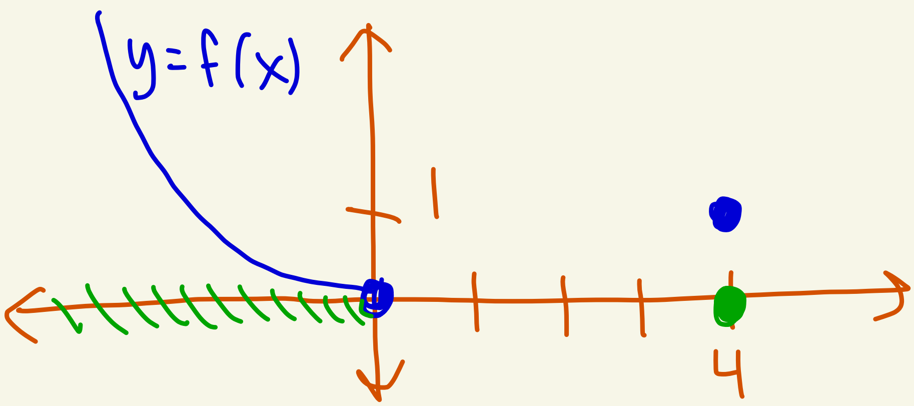
Let's discuss this definition with an example.

Keep the following in mind.

$D = (-\infty, 0] \cup \{4\}$

$f : D \to \mathbb{R}$

$$f(x) = \begin{cases} x^2 & \text{if } x \le 0 \\ 1 & \text{if } x = 4 \end{cases}$$



D is green

f is blue

There are two cases for our definition of continuity.

Case 1: Suppose a is a limit point of D.

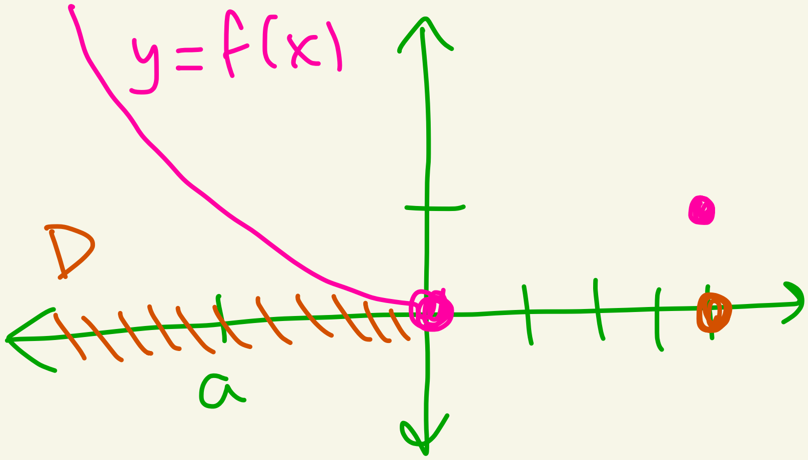
Then we can consider $\lim_{x \to a} f(x)$

By def of continuity, if f is continuous at a, then

① $\lim_{x \to a} f(x)$ exists

② $\lim_{x \to a} f(x) = f(a)$

In our example this is when $a \leq 0$



Case 2: Suppose a is not a limit point of D.

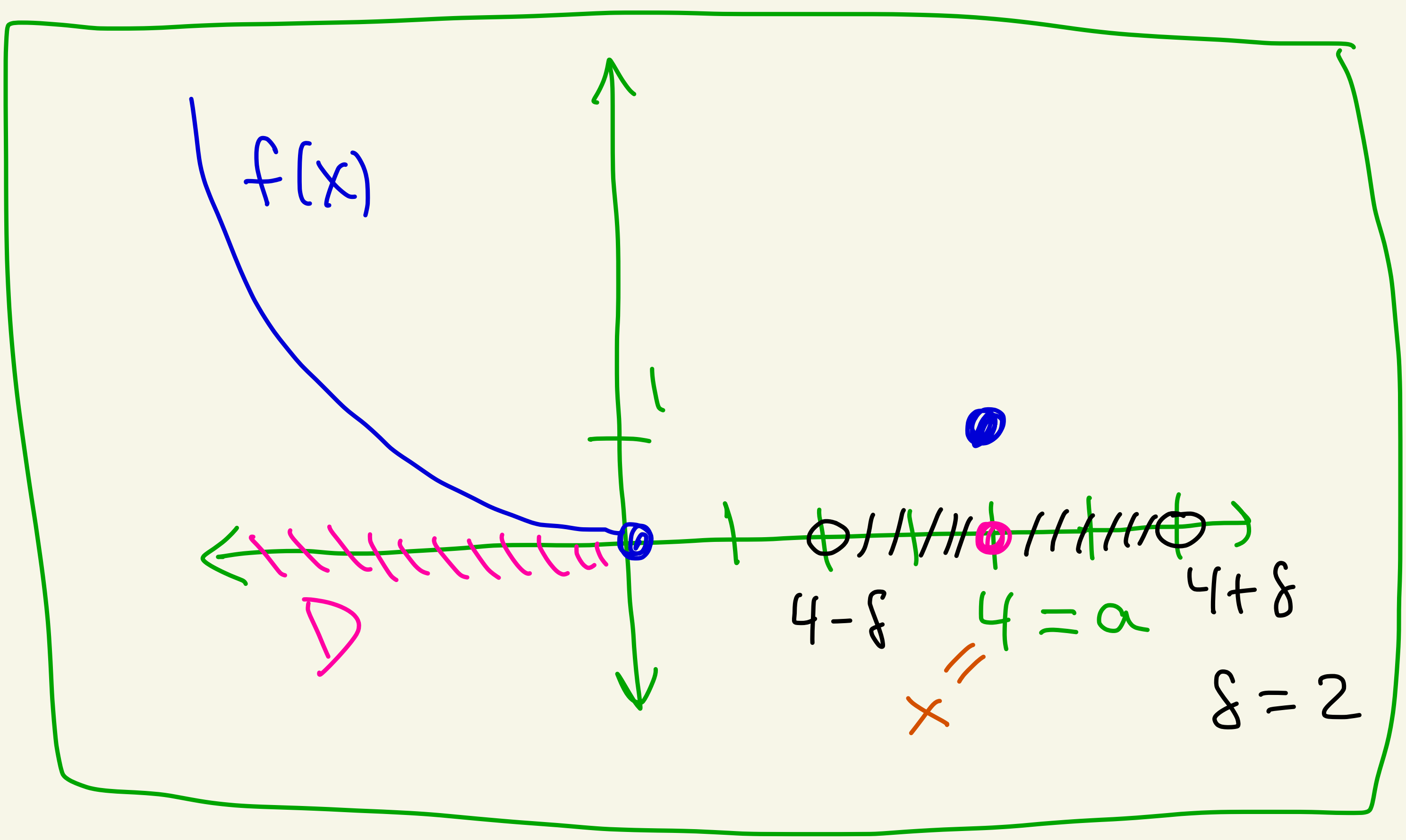
Then there exists $\delta > 0$ where

$$(a-\delta, a+\delta) \cap D = \{a\}$$

Then if $x \in D$ and $|x-a| < \delta$,

we get $|f(x) - f(a)| = |f(a) - f(a)|$
$= 0 < \varepsilon$

for any $\varepsilon > 0$. So, f is continuous at $x = a$.



Ex: Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$.

Let $a \in \mathbb{R}$.

Let's show that f is continuous at a.

$$\boxed{\text{We need to show that if } \varepsilon > 0 \text{ there exists } \delta > 0 \text{ where if } |x-a| < \delta, \text{ then } |x^2 - a^2| < \varepsilon}$$

Let $\varepsilon > 0$.

We have

$$|x^2 - a^2| = |(x-a)(x+a)|$$
$$= \underbrace{|x-a|}_{\substack{\text{we can} \\ \text{control with } \delta}} \cdot \underbrace{|x+a|}_{\substack{\text{pick a} \\ \text{starting} \\ \text{bound on}}}$$

δ to bound this term

Suppose $\delta \leq 1$. $\leftarrow$ arbitrary starting bound

Suppose $|x-a| < \delta \leq 1$.

Then,

$$
\begin{aligned}
|x+a| &= |x-a+a+a| \\
&\leq |x-a| + |2a| \\
&< 1 + 2|a|
\end{aligned}
$$

So if $|x-a| < \delta \leq 1$, then

$$
\begin{aligned}
|x^2-a^2| &= |x-a| \cdot |x+a| \\
&< |x-a| \cdot (1+2|a|)
\end{aligned}
$$

Let $\delta = \min\left\{ \frac{\varepsilon}{1+2|a|}, 1 \right\}$

So, $\delta \le \frac{\varepsilon}{1+2|a|}$ and $\delta \le 1$.

If $|x-a|<\delta$, then

$$|x^2-a^2| = |x-a|\cdot|x+a|$$

$$< |x-a|\cdot(1+2|a|)$$

$\delta \le 1$

$$< \frac{\varepsilon}{1+2|a|}(1+2|a|)$$

$|x-a|<\delta\le\frac{\varepsilon}{1+2|a|}$

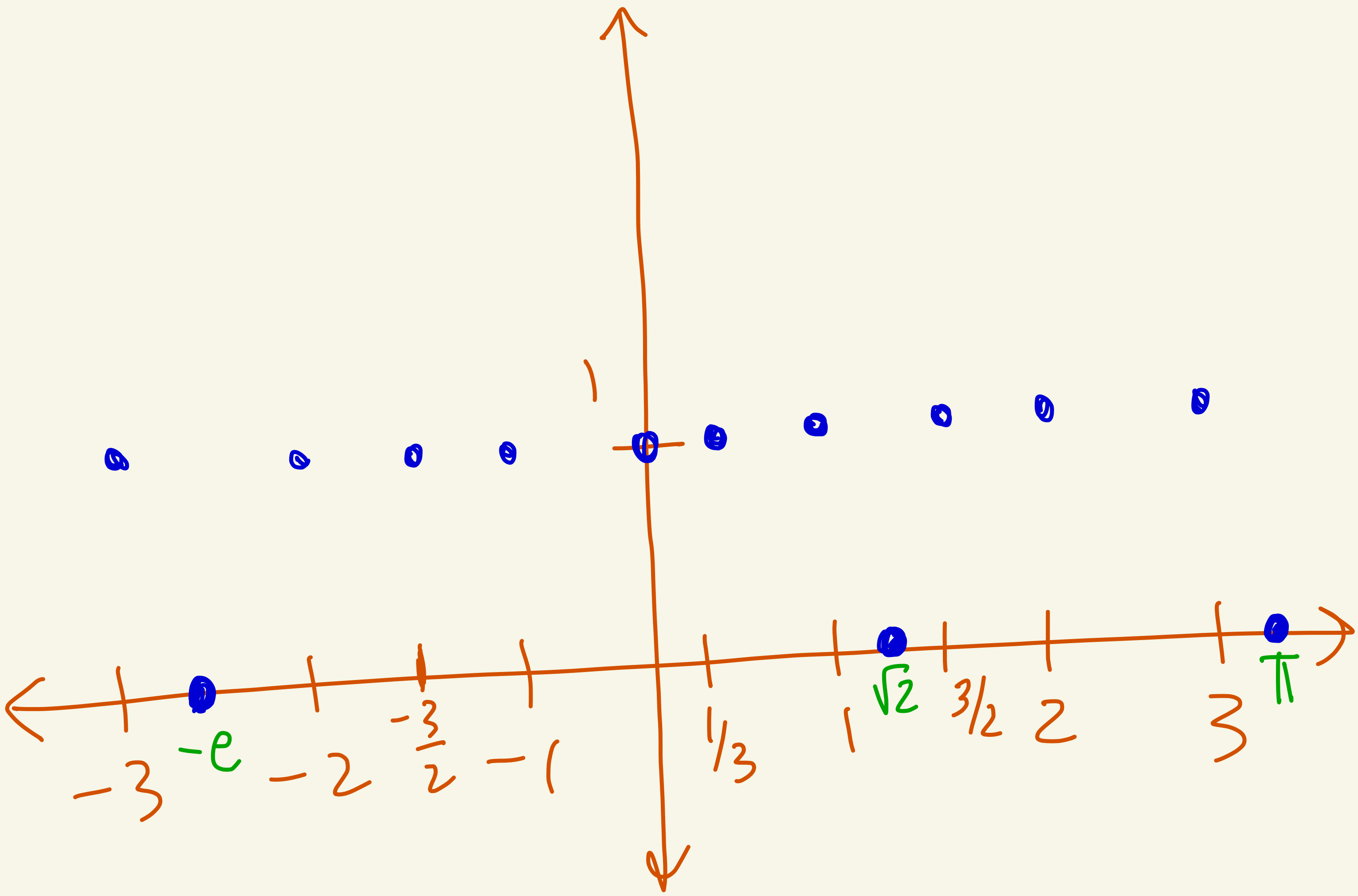
$$= \varepsilon.$$

$\blacksquare$

Ex: (Dirichlet's function - 1829)

Let $f: \mathbb{R} \to \mathbb{R}$ be

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$



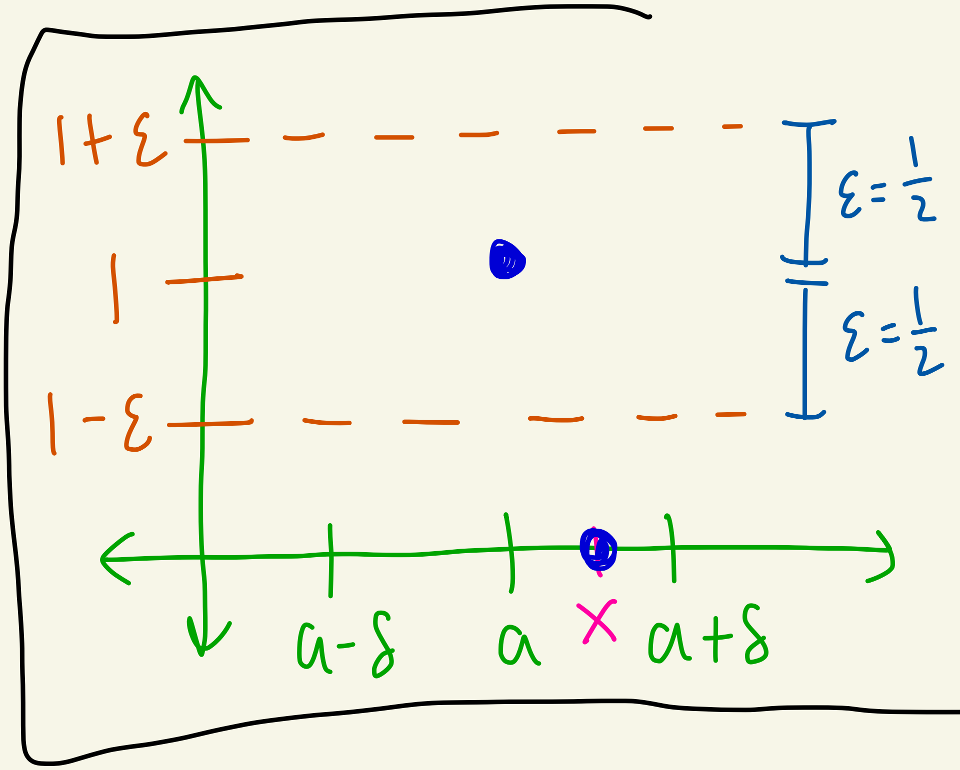
Claim: f is not continuous anywhere.

proof:

Let $a \in \mathbb{R}$.

Let $\varepsilon = 1/2$.

Case 1: Suppose a is rational.



Then, $f(a) = 1$

Given any $\delta > 0$ there exists an irrational number x with $a - \delta < x < a + \delta$

making

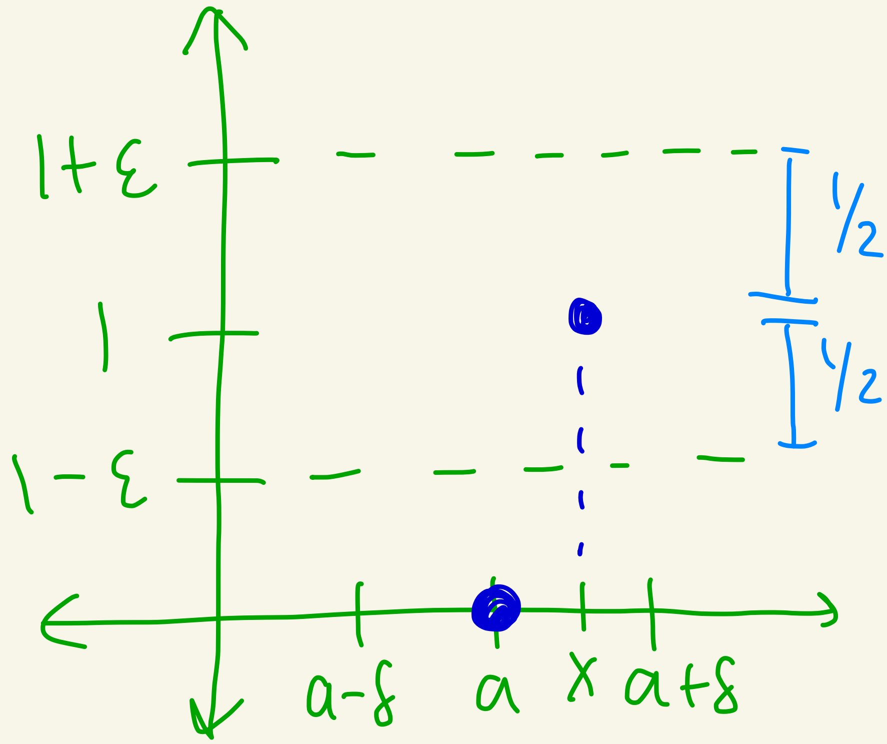
$$|f(x) - f(a)| = |0 - 1| = 1 > \varepsilon$$

So, f is not continuous at a.

Case 2: Suppose a is irrational.

Then, $f(a) = 0$.

Given any δ, there exists a rational number x with $a-\delta < x < a+\delta$.



Then,

$$|f(x) - f(a)| = |1 - 0| = 1 > \varepsilon$$

So, f is not continuous at a.

$\blacksquare$