Math 4650

10/22/25

Function-sequence limit theorem

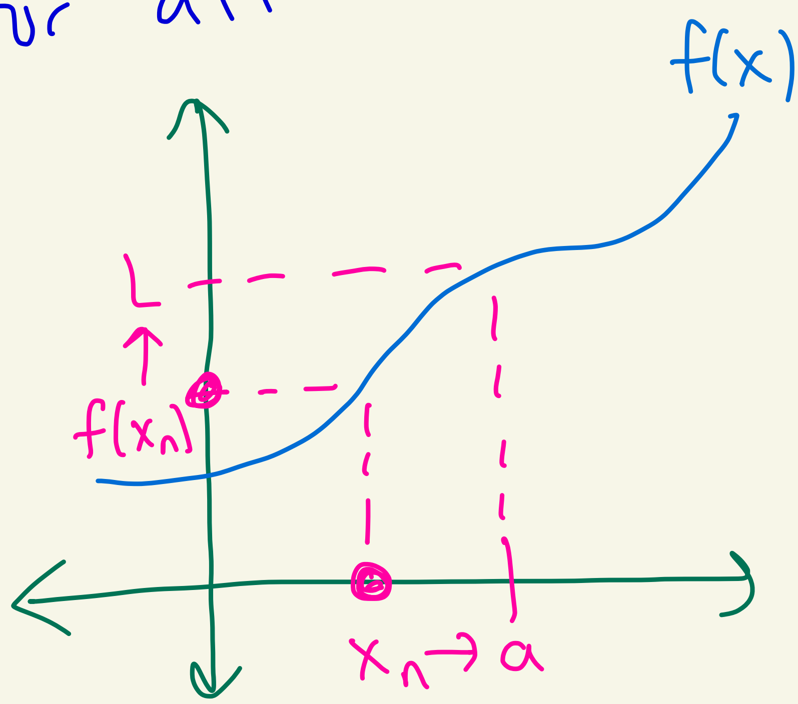
Let $f: D \to \mathbb{R}$ and a is a limit point of D.

The following are equivalent:

① $\lim_{x \to a} f(x) = L$

② $\lim_{n \to \infty} f(x_n) = L$ for every sequence (x_n) contained in D with $x_n \neq a$ for all n and $\underbrace{x_n \to a}$

means: $\lim_{n \to \infty} x_n = a$



Proof:

$\boxed{① \Longrightarrow ②}$: Suppose $\lim_{x \to a} f(x) = L$.

Let (x_n) be a sequence contained in D with $x_n \neq a$ for all n and $\underbrace{x_n \to a}$.

$\lim_{n \to \infty} x_n = a$

Goal: Show $\lim_{n \to \infty} f(x_n) = L$.

Let $\varepsilon > 0$.

Since $\lim_{x \to a} f(x) = L$ there exists $\delta > 0$ where if $0 < |x-a| < \delta$ then $|f(x) - L| < \varepsilon$

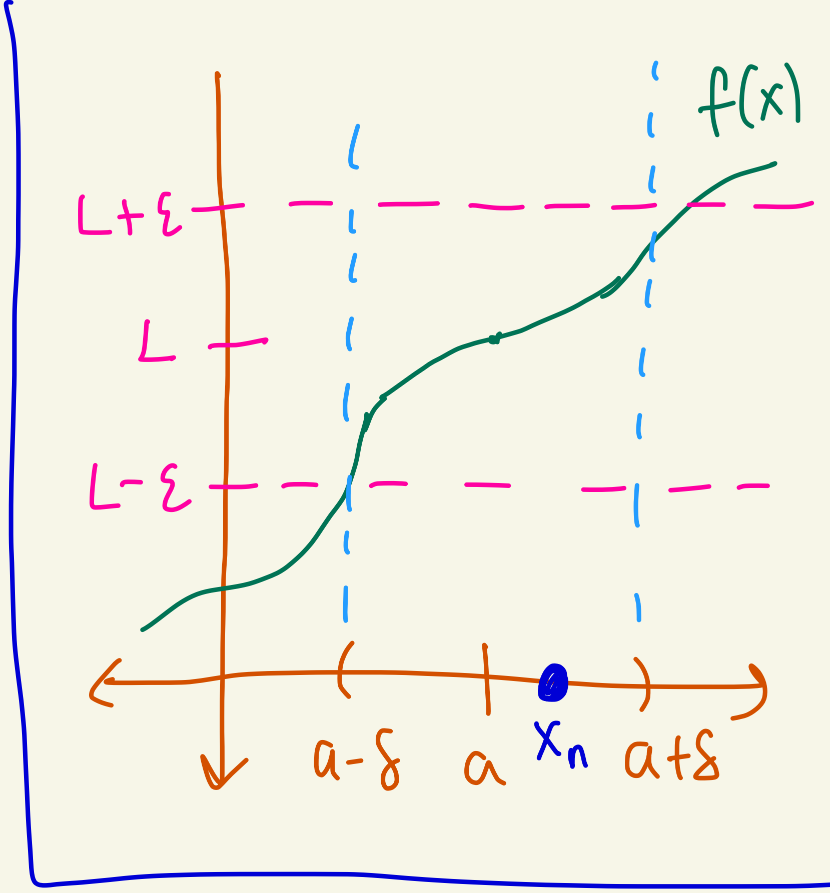
Since $\lim_{n\to\infty} x_n = a$

and $x_n \neq a$ for all n

there exists $N>0$

where if $n \geq N$

then $0 < |x_n - a| < \delta$



So if $n \geq N$,

then $|f(x_n) - L| < \varepsilon$.

So, $\lim_{n\to\infty} f(x_n) = L$.

$\boxed{② \Rightarrow ①}$: Suppose that $\lim_{n\to\infty} f(x_n) = L$

for every sequence (x_n) contained

in D, with $x_n \neq a$ for all n,

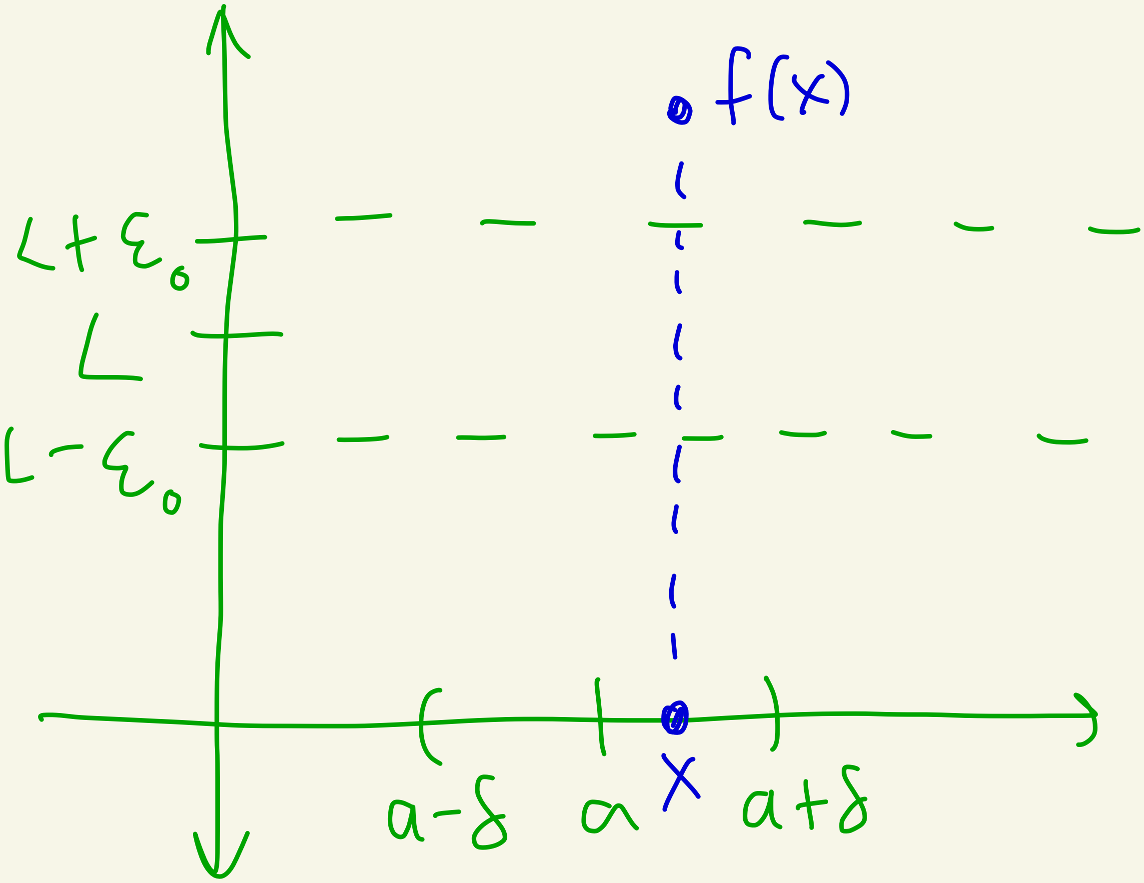
and $\lim_{n\to\infty} x_n = a$.

We must show $\lim_{x \to a} f(x) = L$

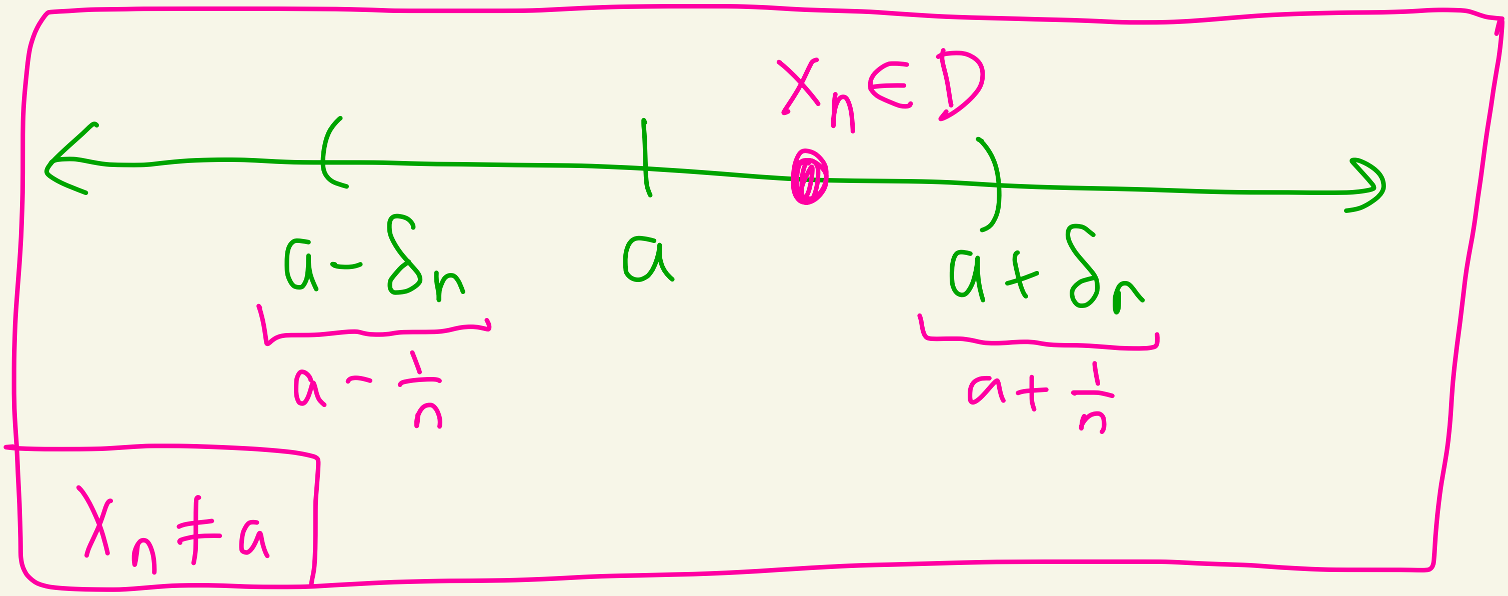
Suppose $\lim_{x \to a} f(x) \neq L$.

Let's show this leads to a contradiction

Since $\lim_{x \to a} f(x) \neq L$, there exists $\varepsilon_0 > 0$ where no matter what $\delta > 0$ you pick, there must exist some $x \in D$ with $0 < |x-a| < \delta$, but $|f(x) - L| \geq \varepsilon_0$.



Let's construct a sequence (x_n) contained in D.
Given $n \geq 1$, set $\delta_n = \frac{1}{n}$.
From above, there will exist $x_n \in D$ with $0 < |x_n - a| < \delta_n$ and $|f(x_n) - L| \geq \varepsilon_0$.



Claim: $\lim_{n\to\infty} x_n = a$

pf: Let $\varepsilon > 0$.
Pick $N > \frac{1}{\varepsilon}$.
Then if $n \geq N$, we get

$$|x_n - a| < \delta_n = \frac{1}{n} \leq \frac{1}{N} < \varepsilon$$

Note also that

$$|f(x_n) - L| \geq \varepsilon_0$$

for all n.

So we've created a sequence (x_n) contained in D, with $x_n \neq a$ for all n, and $x_n \to a$ but $\lim_{n\to\infty} f(x_n) \neq L$.

Contradiction!

Thus, $\lim_{x\to a} f(x) = L$ $\blacksquare$

<u>Theorem:</u> Let $D \subseteq \mathbb{R}$ and a be a limit point of D. Let $f: D \to \mathbb{R}$ and $g: D \to \mathbb{R}$.

Suppose

$$\lim_{x \to a} f(x) = L \quad \text{and} \quad \lim_{x \to a} g(x) = M.$$

Then:

1. $\displaystyle\lim_{x \to a} \left[c f(x)\right] = cL$, where $c \in \mathbb{R}$.
2. $\displaystyle\lim_{x \to a} \left[f(x) + g(x)\right] = L + M$
3. $\displaystyle\lim_{x \to a} \left[f(x) g(x)\right] = LM$
4. $\displaystyle\lim_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \frac{L}{M}$, where

$M \neq 0$ and $g(x) \neq 0$ for all $x \in D$.

<u>Proof:</u> We use the function--sequence theorem.

Let (x_n) be a sequence contained in D, with $x_n \neq a$, and $x_n \to a$.

Since $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, by the function-sequence theorem $\lim_{n \to \infty} f(x_n) = L$ and $\lim_{n \to \infty} g(x_n) = M$

By the algebra of sequences theorem we get that

$$\lim_{n \to \infty} c f(x_n) = c \cdot \lim_{n \to \infty} f(x_n) = cL$$

$$\lim_{n\to\infty}\left[f(x_n)+g(x_n)\right]=\lim_{n\to\infty} f(x_n)+\lim_{n\to\infty} g(x_n)$$
$$=L+M$$

$$\lim_{n\to\infty}\left[f(x_n)\,g(x_n)\right]=\lim_{n\to\infty} f(x_n)\cdot\lim_{n\to\infty} g(x_n)$$
$$=L\cdot M$$

Reapply the function-sequence limit theorem to get 1, 2, 3.

We can do this because (x_n) was arbitrary.

For 4, suppose $M\neq 0$ and $g(x)\neq 0$ for $x\in D$.

Since (x_n) is contained in D we get $g(x_n)\neq 0$ for all n.

By the algebra of sequences thm

$$\lim_{n\to\infty}\left[\frac{f(x_n)}{g(x_n)}\right] = \frac{\lim\limits_{n\to\infty} f(x_n)}{\lim\limits_{n\to\infty} g(x_n)} = \frac{L}{M}$$

Reapply the function-sequence limit theorem to get (4), that is

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{L}{M} \qquad \blacksquare$$