Math 4650

10/20/25

Ex: Let's prove that

$$\lim_{x \to -3} \frac{1}{x+2} = -1$$

Proof:

Let $\varepsilon > 0$.

We need to find $\delta > 0$ where if $0 < |x-(-3)| < \delta$, then $\left| \frac{1}{x+2} - (-1) \right| < \varepsilon$

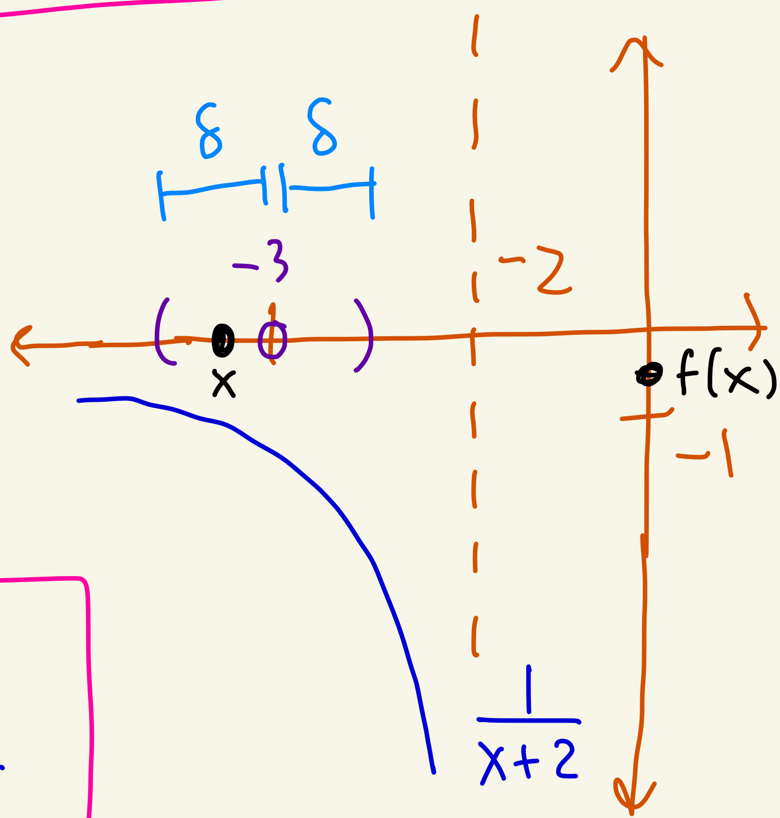
We have that

$$\left| \frac{1}{x+2} - (-1) \right| = \left| \frac{1+x+2}{x+2} \right|$$

$$= \frac{|x+3|}{|x+2|} = \underbrace{|x+3|}_{\text{we can control with } \delta} \cdot \underbrace{\frac{1}{|x+2|}}_{\text{pick a starting bound on } \delta \text{ to deal with this term}}$$

Let $\delta \leq \frac{1}{2}$.

picked to avoid asymptote.



Suppose

$0 < |x-(-3)| < \delta \leq \frac{1}{2}$

Then

$|x+3| < \frac{1}{2}$

Then, $-\frac{1}{2} < x+3 < \frac{1}{2}$.

$|z| < c$ iff $-c < z < c$

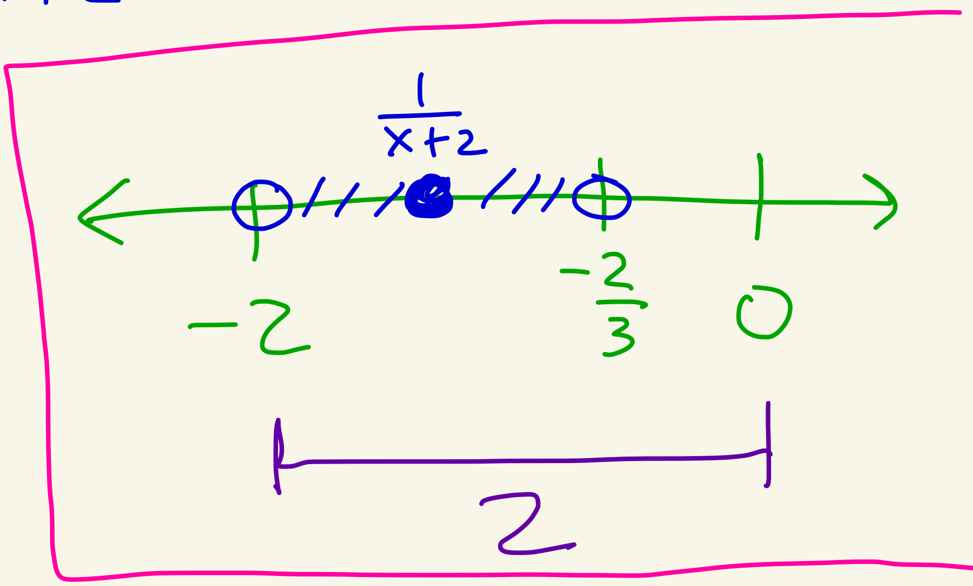
Then, $-1 - \frac{1}{2} < x + 2 < -1 + \frac{1}{2}$

Then, $-\frac{3}{2} < x + 2 < -\frac{1}{2}$

Then, $-2 < \frac{1}{x+2} < -\frac{2}{3}$

Then,

$\left| \frac{1}{x+2} \right| < 2$



So, if $0 < |x - (-3)| < \frac{1}{2}$,

then $\frac{1}{|x+2|} < 2$.

Thus, if $0 < |x - (-3)| < \frac{1}{2}$,

then $\left| \frac{1}{x+2} - (-1) \right| = |x+3| \cdot \frac{1}{|x+2|}$

$< 2 \cdot |x+3|$

Now suppose $\delta = \min\left\{\frac{1}{2}, \frac{\varepsilon}{2}\right\}$.

Then, $\delta \le \frac{1}{2}$ and $\delta \le \frac{\varepsilon}{2}$.

If $0 < |x-(-3)| < \delta$, then

$$\left|\frac{1}{x+2} - (-1)\right| = |x+3| \cdot \frac{1}{|x+2|}$$

$$< 2 \cdot |x+3|$$

$\boxed{\delta \le \frac{1}{2}}$

$$< 2 \cdot \frac{\varepsilon}{2}$$

$\boxed{|x+3| < \delta \le \frac{\varepsilon}{2}}$

$$= \varepsilon.$$

$\blacksquare$

<u>Theorem:</u> Limits are unique.

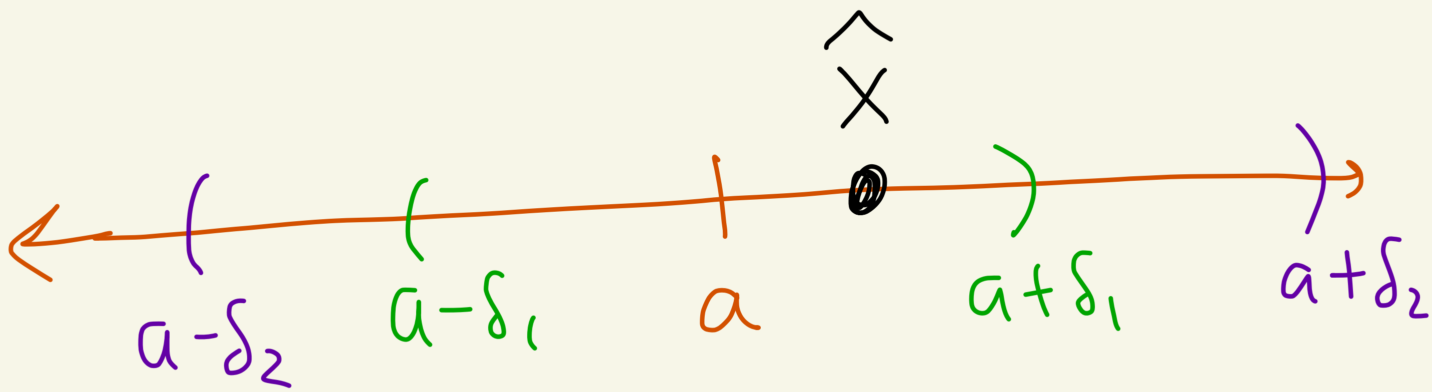
That is, if $f : D \to \mathbb{R}$ and a is a limit point of D and $\lim_{x \to a} f(x) = L_1$ and $\lim_{x \to a} f(x) = L_2$, then $L_1 = L_2$.

<u>proof:</u>

Let $\varepsilon > 0$.

Since $\lim_{x \to a} f(x) = L_1$ there exists $\delta_1 > 0$ where if $0 < |x - a| < \delta_1$ and $x \in D$, then $|f(x) - L_1| < \frac{\varepsilon}{2}$

Since $\lim_{x \to a} f(x) = L_2$ there exists $\delta_2 > 0$ where if $0 < |x-a| < \delta_2$ and $x \in D$, then $|f(x) - L_2| < \frac{\varepsilon}{2}$



Let's assume $\delta_1 \le \delta_2$ for simplicity.

Since a is a limit point of D, there exists $\hat{x} \in D$ with $0 < |\hat{x} - a| < \delta_1 \le \delta_2$.

Then,

$$|L_1 - L_2| = |L_1 - f(\hat{x}) + f(\hat{x}) - L_2|$$

$$\le |L_1 - f(\hat{x})| + |f(\hat{x}) - L_2|$$

$$= |f(\hat{x}) - L_1| + |f(\hat{x}) - L_2|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$= \varepsilon$$

So, $|L_1 - L_2| < \varepsilon$ where ε is an arbitrary positive number.

Thus, $|L_1 - L_2| = 0$.

Then, $L_1 - L_2 = 0$

So, $L_1 = L_2$ $\blacksquare$