Math 4650

10/13/25

$\boxed{\text{HW 2}}$

$\boxed{3(e)}$ Show $\lim_{n\to\infty} \frac{n^2}{2n^2+1} = \frac{1}{2}$

proof:

Let $\varepsilon > 0$.

We have

$$\left| \frac{n^2}{2n^2+1} - \frac{1}{2} \right| = \left| \frac{2n^2 - (2n^2+1)}{(2n^2+1)(2)} \right|$$

$$= \left| \frac{-1}{4n^2+2} \right| = \frac{1}{4n^2+2}$$

We have

$$\frac{1}{4n^2+2} < \varepsilon$$

iff

$$\frac{1}{\varepsilon} < 4n^2+2$$

iff

$$\frac{1}{4\varepsilon} - \frac{1}{2} < n^2$$

iff

$$\sqrt{\frac{1}{4\varepsilon} - \frac{1}{2}} < n$$

Pick $N > \sqrt{\frac{1}{4\varepsilon} - \frac{1}{2}}$

Then, from above, if $n \geq N$, then

$$\left| \frac{n^2}{2n^2+1} - \frac{1}{2} \right| < \varepsilon \qquad \blacksquare$$

$\boxed{\text{HW 2}}$

$\boxed{4(c)}$ Suppose $\lim_{n\to\infty} a_n = A$

and $\lim_{n\to\infty} b_n = B$.

Prove $\lim_{n\to\infty} a_n b_n = AB$.

<u>proof:</u>

Let $\varepsilon > 0$.

We have

$$|a_n b_n - AB| = |a_n b_n - a_n B + a_n B - AB|$$

$$= |a_n(b_n - B) + B(a_n - A)|$$

Δ-inequality

$$\leq |a_n(b_n - B)| + |B(a_n - A)|$$

$$= |a_n||b_n - B| + |B||a_n - A|$$

Since (a_n) converges, it is bounded.

So there exists $M > 0$ where

$|a_n| < M$ for all n.

Since $\lim_{n\to\infty} a_n = A$ there exists

$N_1 > 0$ where if $n \geq N_1$

then $|a_n - A| < \dfrac{\varepsilon}{2(|B|+1)}$

Since $\lim_{n\to\infty} b_n = B$ there exists

$N_2 > 0$ where if $n \geq N_2$

then $|b_n - B| < \dfrac{\varepsilon}{2M}$.

Let $N = \max\{N_1, N_2\}$.

If $n \geq N$, we get

$$
\begin{aligned}
|a_n b_n - AB| &\leq |a_n||b_n - B| + |B||a_n - A| \\
&< M \cdot \frac{\varepsilon}{2M} + |B| \cdot \frac{\varepsilon}{2(|B|+1)} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\left(\frac{|B|}{|B|+1}\right) \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}
$$

So if $n \geq N$, then $|a_n b_n - AB| < \varepsilon$

Thus, $\lim\limits_{n \to \infty} a_n b_n = AB$. $\blacksquare$

Hw 2

5 (Squeeze theorem)

$a_n \leq b_n \leq c_n \quad \forall n$

$a_n \to L$, $c_n \to L$

<u>Prove</u>: $b_n \to L$

Let $\varepsilon > 0$.

Since $\lim_{n\to\infty} a_n = L$ there exists

$N_1 > 0$ where if $n \geq N_1$

then $|a_n - L| < \varepsilon$

So if $n \geq N_1$, then

$-\varepsilon < a_n - L < \varepsilon$

That is, if $n \geq N_1$,

then $L-\varepsilon < a_n < L+\varepsilon$.

Similarly since $\lim_{n\to\infty} c_n = L$

there exists $N_2 > 0$ where

if $n \geq N_2$, then $L-\varepsilon < c_n < L+\varepsilon$.

Let $N = \max\{N_1, N_2\}$

If $n \geq N$, then

$L-\varepsilon < a_n \leq b_n \leq c_n < L+\varepsilon$.

So if $n \geq N$ we have

that $L-\varepsilon < b_n < L+\varepsilon$.

That is if $n \geq N$, then $|b_n - L| < \varepsilon$.

So, $\lim_{n\to\infty} b_n = L$ $\blacksquare$

$\boxed{\text{HW 2}}$

⑦ Suppose $\lim_{n\to\infty} a_n = L$.

We want $M > 0$, $N > 0$ where if $n \geq N$, then $|a_n| \geq M$.

Let $\varepsilon = \frac{|L|}{2}$.

Since $\lim_{n\to\infty} a_n = L$, there exists $N > 0$ where if $n \geq N$ then $|a_n - L| < \frac{|L|}{2}$.

So, if $n \geq N$, then

$$|L| = |L - a_n + a_n| \leq |L - a_n| + |a_n|$$

$$< \frac{|L|}{2} + |a_n|$$

So if $n \geq N$, then

$$|L| < \frac{|L|}{2} + |a_n|$$

$$|L| - \frac{|L|}{2} < |a_n|$$

$$|L|\left(1 - \frac{1}{2}\right) < |a_n|$$

Then if $n \geq N$ we get

$$\frac{|L|}{2} < |a_n|$$

Set $M = \frac{|L|}{2}$.

If $n \geq N$, then $|a_n| > M$.

$\blacksquare$

$\boxed{\text{Hw 1}}$

8 (d)

Let $a, b, x, y \in \mathbb{R}$.

Prove: If $a < x < b$ and $a < y < b$
then $|x-y| < b-a$

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proof:

We have that
$$a < x < b \text{ and } a < y < b.$$

Thus,
$$a < x < b \text{ and } -b < -y < -a.$$

Adding gives
$$a-b < x-y < b-a.$$

Thus, $-(b-a) < x-y < (b-a)$]

Hence

$$|x-y| < b-a.$$

$\blacksquare$

$$-c < z < c$$
iff
$$|z| < c$$