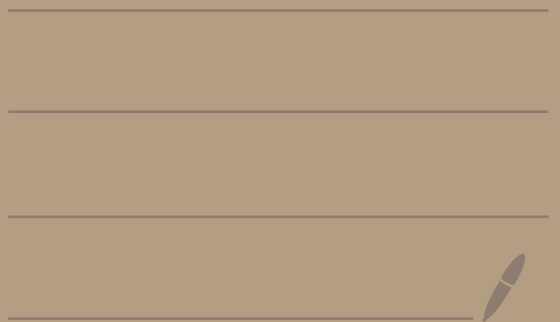


Math 4650

10/1/25



Topic 3 continued...

Comparison test

Suppose that

$$0 \leq a_n \leq b_n$$

for all $n \geq K$, where K is some constant.

Then:

① If $\sum b_n$ converges,
then $\sum a_n$ converges.

② If $\sum a_n$ diverges,
then $\sum b_n$ diverges.

Contrapositive
of
each
other

Proof: Suppose $\sum b_n$ converges

Let S_k be the k -th partial sum of $\sum b_n$ and t_k be the k -th partial sum of $\sum a_n$.

So,

$$S_k = b_1 + b_2 + \dots + b_k$$

$$t_k = a_1 + a_2 + \dots + a_k$$

Since $\sum b_n$ converges, we know $(S_k)_{k=1}^{\infty}$ converges.

Thus, $(S_k)_{k=1}^{\infty}$ is a Cauchy sequence.

Let $\varepsilon > 0$.

Then, there exists $N > K$ where if $m \geq n \geq N$ then

$$|S_m - S_n| < \varepsilon.$$

do this so
 $a_n \leq b_n$

Then if $m \geq n \geq N$, we get

$$\begin{aligned} & |t_m - t_n| \\ &= |(a_1 + a_2 + \dots + a_m) - (a_1 + a_2 + \dots + a_n)| \\ &= |a_{n+1} + a_{n+2} + \dots + a_m| \end{aligned}$$

↑

$$\begin{aligned} & |(a_1 + a_2 + a_3 + a_4) - (a_1 + a_2)| \\ &= |a_3 + a_4| \end{aligned} \quad \begin{array}{l} m=4 \\ n=2 \end{array}$$

↙ $a_i \geq 0$

$$= a_{n+1} + a_{n+2} + \dots + a_m$$

$$\leq b_{n+1} + b_{n+2} + \dots + b_m$$

↑ $n, m \geq N > K$

$$= |b_{n+1} + b_{n+2} + \dots + b_m|$$

↑

$$b_i \geq 0$$

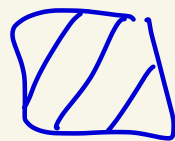
$$= |(b_1 + b_2 + \dots + b_m) - (b_1 + b_2 + \dots + b_n)|$$

$$= |s_m - s_n| < \varepsilon$$

since
 $m \geq n \geq N$

Thus, $(t_k)_{k=1}^{\infty}$ is a Cauchy sequence and thus converges.

So, $\sum a_n$ converges.



Ex: Consider

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + n} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \frac{1}{42} + \dots$$

We have

$$\frac{1}{n^2+n} < \frac{1}{n^2}$$

when $n \geq 1$.

We know $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

($p=2$ series, $p > 1$).

By the comparison
test, $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ converges.

Ex: (p -series, $0 < p < 1$)

Suppose $0 < p < 1$.

If $n \geq 1$, then $n^p \leq n$.

$$\text{So, } \frac{1}{n} \leq \frac{1}{n^p}.$$

$$\left. \begin{array}{l} p = 1/2 \\ n = 5 \\ \sqrt{5} \leq 5 \end{array} \right\} \textcircled{\text{Ex}}$$

We know the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

So, by the comparison test,
 $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges ($0 < p < 1$)

Theorem (Alternating series test)

Let (a_n) be monotonically decreasing and $a_n > 0$ for all n . Also assume $\lim_{n \rightarrow \infty} a_n = 0$.

Then,

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \dots$$

Converges.

Ex: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$

$$a_n = \frac{1}{n} > 0$$

$$a_n = \frac{1}{n} > \frac{1}{n+1} = a_{n+1}$$

monotonically decreasing

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

Thus, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges

called
alternating
harmonic
series

by the alternating series test

proof of alternating series test:

Let S_k be the k -th partial sum of the series.

Here

$$S_k = a_1 - a_2 + a_3 - a_4 + \dots + (-1)^{k+1} a_k$$

We must show (S_k) converges.

We are given that $a_n \geq a_{n+1}$
for all n .

So, $a_n - a_{n+1} \geq 0$ for all n .

Let's look at S_{2k} .

$$\begin{aligned} \text{We have } S_{2k} &= \overbrace{(a_1 - a_2)}^{\geq 0} + \overbrace{(a_3 - a_4)}^{\geq 0} \\ &\quad + \dots + \overbrace{(a_{2k-1} - a_{2k})}^{\geq 0} \\ &\leq (a_1 - a_2) + (a_3 - a_4) \\ &\quad + \dots + (a_{2k-1} - a_{2k}) \\ &\quad + \underbrace{(a_{2k+1} - a_{2k+2})}_{\geq 0} = S_{2k+2} \end{aligned}$$

That is, $(S_{2k})_{k=1}^{\infty}$ is $\left. \begin{array}{l} \text{monotonically increasing.} \\ = S_{2(k+1)} \end{array} \right\}$

That is,

$$S_2, S_4, S_6, S_8, S_{10}, \dots$$

is monotonically increasing.

Also,

$$S_{2k} = a_1 - \overbrace{(a_2 - a_3)}^{\geq 0} - \overbrace{(a_4 - a_5)}^{\geq 0} - \dots - \overbrace{(a_{2k-2} - a_{2k-1})}^{\geq 0} - \underbrace{a_{2k}}_{> 0}$$

$$\text{So, } S_{2k} < a_1.$$

Thus, $(S_{2k})_{k=1}^{\infty}$ is monotonically increasing and bounded.

Thus, by the monotone convergence theorem, (S_{2k}) converges to some real number L .

Let $\varepsilon > 0$.

Since $\lim_{k \rightarrow \infty} S_{2k} = L$, there

exists $N_1 > 0$ where if
 $k \geq N_1$ then $|S_{2k} - L| < \frac{\varepsilon}{2}$.

Since $\lim_{k \rightarrow \infty} a_k = 0$ there

exists $N_2 > 0$ where if
 $k \geq N_2$ then

$$|a_{2k+1}| = |a_{2k+1} - 0| < \varepsilon/2.$$

If $k \geq \max \{N_1, N_2\}$, then

$$\begin{aligned} |S_{2k+1} - L| &= |S_{2k} + a_{2k+1} - L| \\ &\leq |S_{2k} - L| + |a_{2k+1}| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Thus, if $k \geq \max \{N_1, N_2\}$
then we get both

$$|S_{2k} - L| < \frac{\varepsilon}{2} < \varepsilon \quad \text{and} \quad |S_{2k+1} - L| < \varepsilon$$

Hence if $m \geq 2 \max \{N_1, N_2\} + 1$
then $|S_m - L| < \varepsilon$.

Thus, (S_m) converges.

