Math 4650
10/1/25

Topic 3 continued...

Comparison test

Suppose that

$$0 \leq a_n \leq b_n$$

for all $n \geq K$, where K is some constant.

Then:

① If $\sum b_n$ converges, then $\sum a_n$ converges.

② If $\sum a_n$ diverges, then $\sum b_n$ diverges.

contrapositive of each other

proof: Suppose $\sum b_n$ converges

Let s_k be the k-th partial sum of $\sum b_n$ and t_k be the k-th partial sum of $\sum a_n$.

So,

$$s_k = b_1 + b_2 + \dots + b_k$$

$$t_k = a_1 + a_2 + \dots + a_k$$

Since $\sum b_n$ converges, we know $(s_k)_{k=1}^{\infty}$ converges.

Thus, $(s_k)_{k=1}^{\infty}$ is a Cauchy sequence.

Let $\varepsilon > 0$.

Then, there exists $\overbrace{N > K}^{\text{do this so } a_n \le b_n}$ where if $m \geq n \geq N$ then

$$|s_m - s_n| < \varepsilon.$$

Then if $m \geq n \geq N$, we get

$$|t_m - t_n|$$

$$= |(a_1 + a_2 + \dots + a_m) - (a_1 + a_2 + \dots + a_n)|$$

$$= |a_{n+1} + a_{n+2} + \dots + a_m|$$

$$|(a_1 + a_2 + a_3 + a_4) - (a_1 + a_2)|$$

$$= |a_3 + a_4|$$

$m = 4$
$n = 2$

$a_i \geq 0$

$$= a_{n+1} + a_{n+2} + \dots + a_m$$

$$\leq b_{n+1} + b_{n+2} + \dots + b_m$$

$\boxed{n, m \geq N > K}$

$$= |b_{n+1} + b_{n+2} + \dots + b_m|$$

$b_i \geq 0$

$$= \left|(b_1 + b_2 + \cdots + b_m) - (b_1 + b_2 + \cdots + b_n)\right|$$

$$= |s_m - s_n| < \varepsilon$$

since $m \geq n \geq N$

Thus, $(t_k)_{k=1}^{\infty}$ is a Cauchy sequence and thus converges,

So, $\sum a_n$ converges. $\blacksquare$

<u>Ex:</u> Consider

$$\sum_{n=1}^{\infty} \frac{1}{n^2+n} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \frac{1}{42} + \cdots$$

We have

$$\frac{1}{n^2+n} < \frac{1}{n^2}$$

when $n \geq 1$.

We know $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

($p = 2$ series, $p > 1$).

By the comparison test, $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ converges.

<u>Ex:</u> (p-series, $0 < p < 1$)

Suppose $0 < p < 1$.

If $n \geq 1$, then $n^p \leq n$.

So, $\frac{1}{n} \leq \frac{1}{n^p}$.

$$\boxed{\left.\begin{matrix} p = 1/2 \\ n = 5 \\ \sqrt{5} \leq 5 \end{matrix}\right\} \text{Ex}}$$

We know the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

So, by the comparison test,

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ diverges } (0 < p < 1)$$

<u>Theorem</u> (Alternating series test)

Let (a_n) be monotonically decreasing and $a_n > 0$ for all n. Also assume $\lim_{n \to \infty} a_n = 0$.

Then,

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \ldots$$

converges.

<u>Ex:</u> $$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

$$a_n = \frac{1}{n} > 0$$

$$a_n = \frac{1}{n} > \frac{1}{n+1} = a_{n+1}$$ ← monotonically decreasing

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

Thus, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges by the alternating series test

called alternating harmonic series

<u>proof of alternating series test:</u>

Let S_k be the k-th partial sum of the series.

Here

$$S_k = a_1 - a_2 + a_3 - a_4 + \dots + (-1)^{k+1} a_k$$

We must show (S_k) converges.

We are given that $a_n \geq a_{n+1}$ for all n.

So, $a_n - a_{n+1} \geq 0$ for all n.

Let's look at S_{2k}.

We have

$$S_{2k} = \overbrace{(a_1 - a_2)}^{\geq 0} + \overbrace{(a_3 - a_4)}^{\geq 0} + \dots + \overbrace{(a_{2k-1} - a_{2k})}^{\geq 0}$$

$$\leq (a_1 - a_2) + (a_3 - a_4) + \dots + (a_{2k-1} - a_{2k}) + \underbrace{(a_{2k+1} - a_{2k+2})}_{\geq 0} = S_{2k+2}$$

$= s_{2(k+1)}$

That is, $(s_{2k})_{k=1}^{\infty}$ is monotonically increasing.

That is,

$$s_2, s_4, s_6, s_8, s_{10}, \ldots$$

is monotonically increasing.

Also,

$$s_{2k} = a_1 - \overbrace{(a_2 - a_3)}^{\geq 0} - \overbrace{(a_4 - a_5)}^{\geq 0} - \ldots - \overbrace{(a_{2k-2} - a_{2k-1})}^{\geq 0} - \underbrace{a_{2k}}_{>0}$$

So, $s_{2k} < a_1$.

Thus, $(s_{2k})_{k=1}^{\infty}$ is monotonically increasing and bounded.

Thus, by the monotone convergence theorem, (s_{2k}) converges to some real number L.

Let $\varepsilon > 0$.

Since $\lim_{k \to \infty} s_{2k} = L$, there exists $N_1 > 0$ where if $k \geq N_1$ then $|s_{2k} - L| < \frac{\varepsilon}{2}$.

Since $\lim_{k \to \infty} a_k = 0$ there exists $N_2 > 0$ where if $k \geq N_2$ then

$|a_{2k+1}| = |a_{2k+1} - 0| < \varepsilon/2$.

If $k \geq \max\{N_1, N_2\}$, then

$$|S_{2k+1} - L| = |S_{2k} + a_{2k+1} - L|$$

$$\leq |S_{2k} - L| + |a_{2k+1}|$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus, if $k \geq \max\{N_1, N_2\}$ then we get both

$|S_{2k} - L| < \frac{\varepsilon}{2} < \varepsilon$ and $|S_{2k+1} - L| < \varepsilon$

Hence if $m \geq 2\max\{N_1, N_2\} + 1$ then $|S_m - L| < \varepsilon$.

Thus, (S_m) converges. $\blacksquare$
