Def: Let V be a vector space over a field F.

Let $v_1, v_2, \ldots, v_n$ be in V.

- We say that $v_1, v_2, \ldots, v_n$ are linearly dependent if there exist $c_1, c_2, \ldots, c_n \in F$, that are not all zero, where

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = \vec{0}$$

- If there are no such c_i's then the vectors $v_1, v_2, \ldots, v_n$ are linearly independent

Another way:

Given $v_1, v_2, \ldots, v_n \in V$.

There is always at least one solution to

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = \vec{0} \qquad (*)$$

That is, $\boxed{c_1 = 0, c_2 = 0, \ldots, c_n = 0.}$

If that's the only solution, then $v_1, v_2, \ldots, v_n$ are <u>lin. ind.</u>

If there are more solutions then $v_1, v_2, \ldots, v_n$ are <u>lin. dep.</u>

Ex: Let $V = \mathbb{R}^3$ and $F = \mathbb{R}$.

Let $v_1 = (1,0,1)$, $v_2 = (-1,2,1)$, $v_3 = (0,2,2)$

Are v_1, v_2, v_3 linearly independent or dependent?

We want to solve

$$\underbrace{c_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}_{c_1 v_1 + c_2 v_2 + c_3 v_3 = \vec{0}}$$

This gives $\begin{pmatrix} c_1 \\ 0 \\ c_1 \end{pmatrix} + \begin{pmatrix} -c_2 \\ 2c_2 \\ c_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 2c_3 \\ 2c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

So, $\begin{pmatrix} c_1 - c_2 \\ 2c_2 + 2c_3 \\ c_1 + c_2 + 2c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

This gives a system.

$$\boxed{\begin{aligned} c_1 - c_2 \quad &= 0 \\ 2c_2 + 2c_3 &= 0 \\ c_1 + c_2 + 2c_3 &= 0 \end{aligned}}$$

$$\left(\begin{array}{ccc|c} \boxed{1} & -1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 1 & 2 & 0 \end{array}\right) \xrightarrow{-R_1+R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & \boxed{2} & 2 & 0 \\ 0 & 2 & 2 & 0 \end{array}\right)$$

make these 0's

make 1

$$\xrightarrow{\frac{1}{2}R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & \boxed{2} & 2 & 0 \end{array}\right) \xrightarrow{-2R_2+R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

make 0

$$\begin{array}{rcccc|cl} & (0 & -2 & -2 & & 0) & \leftarrow -2R_2 \\ + & (0 & 2 & 2 & & 0) & \leftarrow R_3 \\ \hline & (0 & 0 & 0 & & 0) & \leftarrow -2R_2+R_3 \end{array}$$

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This becomes

$$
\begin{aligned}
\boxed{c_1} - c_2 &= 0 \quad (1)\\
\boxed{c_2} + c_3 &= 0 \quad (2)\\
0 &= 0 \quad (3)
\end{aligned}
$$

leading variables: c_1, c_2

free variables: c_3

(free variables are the ones that aren't leading variables)

Solve for leading variables

Give free variables new name.

$$
\begin{aligned}
c_1 &= c_2 \quad (1)\\
c_2 &= -c_3 \quad (2)
\end{aligned}
$$

$$c_3 = t$$

(You don't need $0=0$ eqn (3))

Next back-substitute.

$c_3 = t$

② gives $c_2 = -c_3 = -t$

① gives $c_1 = c_2 = -t$

So, the solutions to the system are

$$\boxed{\begin{array}{l} c_1 = -t \\ c_2 = -t \\ c_3 = t \\ \text{where } t \text{ can be any} \\ \quad \text{number from } F = \mathbb{R} \end{array}}$$

Thus,

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = \vec{0}$$

becomes

$$\boxed{-t v_1 - t v_2 + t v_3 = \vec{0}}$$

For ex: set $t=1$ to get

$$(-1)v_1 + (-1)v_2 + (1)v_3 = \vec{0} \leftarrow \boxed{v_3 = v_1 + v_2}$$

Thus, v_1, v_2, v_3 are <u>lin. dep.</u>

$\boxed{= v_1 + v_2}$

Ex: Let $V = P_2(\mathbb{R})$, $F = \mathbb{R}$

Let $w_1 = -3 + 4x^2$

$w_2 = 5 - x + 2x^2$

$w_3 = 1 + x + 3x^2$

Are w_1, w_2, w_3 lin. dep. or lin. ind.?

We want to solve:

$$\underbrace{c_1(-3+4x^2) + c_2(5-x+2x^2) + c_3(1+x+3x^2) = 0 + 0x + 0x^2}_{c_1 w_1 + c_2 w_2 + c_3 w_3 = \vec{0}}$$

This gives

$$-3c_1 + 4c_1x^2 + 5c_2 - c_2x + 2c_2x^2 + c_3 + c_3x + 3c_3x^2 = 0 + 0x + 0x^2$$

This becomes

$$(-3c_1 + 5c_2 + c_3) + (-c_2 + c_3)x + (4c_1 + 2c_2 + 3c_3)x^2 = 0 + 0x + 0x^2$$

This gives

$$\boxed{\begin{aligned} -3c_1 + 5c_2 + c_3 &= 0 \\ -c_2 + c_3 &= 0 \\ 4c_1 + 2c_2 + 3c_3 &= 0 \end{aligned}}$$

Let's solve it!

make into 1

$$\left(\begin{array}{ccc|c} \boxed{-3} & 5 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 4 & 2 & 3 & 0 \end{array}\right) \xrightarrow{R_3 + R_1 \to R_1} \left(\begin{array}{ccc|c} \boxed{1} & 7 & 4 & 0 \\ \boxed{0} & -1 & 1 & 0 \\ \boxed{4} & 2 & 3 & 0 \end{array}\right)$$

make 0

make 1

$$\xrightarrow{-4R_1 + R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & 7 & 4 & 0 \\ 0 & \boxed{-1} & 1 & 0 \\ 0 & -26 & -13 & 0 \end{array}\right) \xrightarrow{-R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 7 & 4 & 0 \\ 0 & \boxed{1} & -1 & 0 \\ 0 & \boxed{-26} & -13 & 0 \end{array}\right)$$

make 0

$$\xrightarrow{26R_2 + R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & 7 & 4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -39 & 0 \end{array}\right) \xrightarrow{-\frac{1}{39}R_3 \to R_3} \underbrace{\left(\begin{array}{ccc|c} 1 & 7 & 4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)}_{\text{reduced form}}$$

This gives

$$\begin{aligned} \boxed{c_1} + 7c_2 + 4c_3 &= 0 \\ \boxed{c_2} - c_3 &= 0 \\ \boxed{c_3} &= 0 \end{aligned}$$

leading variables: c_1, c_2, c_3

free variables: none

Solve for leading:

$$\boxed{\begin{aligned} c_1 &= -7c_2 - 4c_3 \\ c_2 &= c_3 \\ c_3 &= 0 \end{aligned}} \quad \begin{aligned} &(1) \\ &(2) \\ &(3) \end{aligned}$$

Back substitute:

(3) $c_3 = 0.$

(2) $c_2 = c_3 = 0.$

(1) $c_1 = -7c_2 - 4c_3 = -7(0) - 4(0) = 0$

Thus, the only solution to

$$c_1 w_1 + c_2 w_2 + c_3 w_3 = \vec{0}$$

is

$$c_1 = 0, c_2 = 0, c_3 = 0.$$

Thus,

w_1, w_2, w_3

are linearly independent.

$\boxed{\text{Example}}$

Def: Let V be a vector space over a field F.

Let $v_1, v_2, \dots, v_n$ be in V.

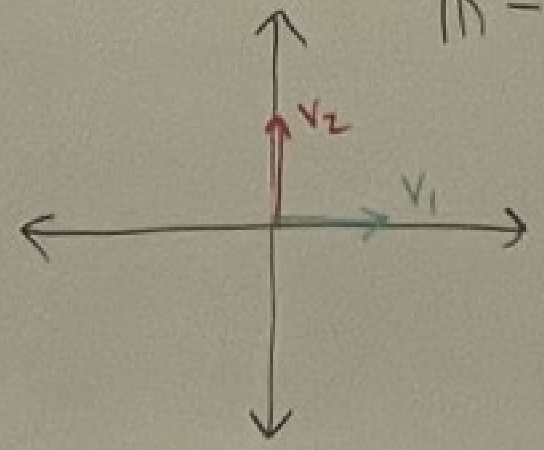
We say that $v_1, v_2, \dots, v_n$ are a basis for V if

① $v_1, v_2, \dots, v_n$ are linearly independent

and ② $V = \text{span}(\{v_1, v_2, \dots, v_n\})$

Ex: Let $V = \mathbb{R}^2$, $F = \mathbb{R}$

$\mathbb{R}^2 = V$



Then $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

is a basis for $\mathbb{R}^2$.

- We already showed previously that $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ span $\mathbb{R}^2$.
- We just have to show that $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are linearly independent.

Suppose $c_1 v_1 + c_2 v_2 = \vec{0}$

Then, $c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Thus, $\begin{pmatrix} c_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

This gives $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

This gives $c_1 = 0$, $c_2 = 0$.

Thus, v_1, v_2 are lin. ind.

So, $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is a basis for $V = \mathbb{R}^2$ over $F = \mathbb{R}$.

Ex: Let $V = M_{2,2}(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a,b,c,d \in \mathbb{R} \right\}$

and $F = \mathbb{R}$.

Let $v_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $v_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Then v_1, v_2, v_3, v_4 is a basis for $M_{2,2}(\mathbb{R})$.

- Let's check spanning.

Pick any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2,2}(\mathbb{R})$.

Then,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= av_1 + bv_2 + cv_3 + dv_4$$

Thus, any vector $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in $\text{span}(\{v_1, v_2, v_3, v_4\})$.

- Next time lin. ind.....