Math 4570
11/30/22

Summary from last time

$T: P_2(\mathbb{R}) \to P_2(\mathbb{R}) \quad T(f) = f'$

$f_T(\lambda) = -\lambda^3$ ← $\boxed{\lambda = 0 \text{ is only eigenvalue}}$

$E_0(T) = \text{span}(\{1\})$

$\beta = [1]$ is an ordered basis for $E_0(T)$

Eigenvalue	algebraic multiplicity	basis for $E_\lambda(T)$	geometric multiplicity
$\lambda = 0$	3	[1]	1

$\boxed{\text{1 vector in basis for } E_0(T)}$ $\quad$ $\boxed{\dim(E_0(T)) = 1}$

Is T diagonalizable? We would need a basis consisting of 3 eigenvectors for $P_2(\mathbb{R})$ to diagonalize T.

No! T is not diagonalizable.

<u>Lemma:</u> Let $T: V \to V$ be a linear transformation where V is a vector space over a field F. Let $v_1, v_2, \ldots, v_r$ be eigenvectors of T with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r$ where $\lambda_i \neq \lambda_j$ if $i \neq j$.

Then $v_1, v_2, \ldots, v_r$ are linearly independent.

This is saying that eigenvectors from different eigenspaces are linearly independent

<u>proof by induction:</u> Let's induce this!

We induct on r.

<u>Base case:</u> Suppose $r = 1$.

So we have one eigenvector v_1 with eigenvalue λ_1

Since v_1 is an eigenvector, we know $v_1 \neq \vec{0}$.

By HW 2 #6, $\{v_1\}$ is a linearly independent set of vectors.

Induction hypothesis Suppose any k eigenvectors of T with distinct eigenvalues are linearly independent.

Now prove for $k+1$: Suppose $v_1, v_2, \dots, v_k, v_{k+1}$ are eigenvectors with corresponding eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k, \lambda_{k+1}$ where $\lambda_i \neq \lambda_j$ if $i \neq j$.

Consider the equation

$$\boxed{c_1 v_1 + c_2 v_2 + \dots + c_k v_k + c_{k+1} v_{k+1} = \vec{0}} \quad (*)$$

where $c_1, c_2, \dots, c_{k+1} \in F$.

We must show $c_1 = 0, c_2 = 0, \dots, c_{k+1} = 0$ is the only solution to $(*)$.

Apply T to $(*)$ and use $T(v_i) = \lambda_i v_i$ and $T(\vec{0}) = \vec{0}$ to get that

$$\boxed{c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + \dots + c_k \lambda_k v_k + c_{k+1} \lambda_{k+1} v_{k+1} = \vec{0}} \quad (**)$$

Multiplying $(*)$ by λ_{k+1} gives

$$\boxed{c_1 \lambda_{k+1} v_1 + c_2 \lambda_{k+1} v_2 + \dots + c_k \lambda_{k+1} v_k + c_{k+1} \lambda_{k+1} v_{k+1} = \vec{0}} \quad (***)$$

$(**) - (***)$ gives $(****)$

$$\boxed{c_1(\lambda_1 - \lambda_{k+1})v_1 + c_2(\lambda_2 - \lambda_{k+1})v_2 + \dots + c_k(\lambda_k - \lambda_{k+1})v_k = \vec{0}}$$

By the induction hypothesis $v_1, v_2, \dots, v_k$ are linearly independent, and ~~therefore~~ ~~thus~~ hence we have

$$c_1(\lambda_1 - \lambda_{k+1}) = 0$$
$$c_2(\lambda_2 - \lambda_{k+1}) = 0$$
$$\vdots \qquad \vdots$$
$$c_k(\lambda_k - \lambda_{k+1}) = 0$$

Since $\lambda_1 - \lambda_{k+1} \neq 0$, $\lambda_2 - \lambda_{k+1} \neq 0, \dots, \lambda_k - \lambda_{k+1} \neq 0$ we must have $\boxed{c_1 = 0, c_2 = 0, \dots, c_k = 0.}$

Plug this back into $(*)$ to get that

$$c_{k+1} v_{k+1} = \vec{0}$$

Since v_{k+1} is an eigenvector, $v_{k+1} \neq \vec{0}$.

Thus, $\boxed{c_{k+1} = 0}$

So, the only solution to $(*)$ is $c_1 = c_2 = \dots = c_{k+1} = 0$.

So, $v_1, v_2, \dots, v_{k+1}$ are lin. ind. By induction we're done $\boxtimes$

Theorem: Let V be a finite-dimensional vector space over a field F. Let $T: V \to V$ be a linear transformation.

Let $n = \dim(V)$.

Let $\lambda_1, \lambda_2, \ldots, \lambda_r$ be the distinct eigenvalues of T.

Let $n_1, n_2, \ldots, n_r$ be the geometric multiplicities of the eigenvalues, that is $n_i = \dim(E_{\lambda_i}(T))$.

For each i, let

$\beta_i = [v_{i,1}, v_{i,2}, \ldots, v_{i,n_i}]$ be an ordered basis for $E_{\lambda_i}(T)$.

Let

$$\begin{aligned}
\beta &= \beta_1 \cup \beta_2 \cup \cdots \cup \beta_r \\
&= [v_{1,1}, v_{1,2}, \ldots, v_{1,n_1}, && \leftarrow \text{basis for } E_{\lambda_1}(T) \\
&\quad v_{2,1}, v_{2,2}, \ldots, v_{2,n_2}, && \leftarrow \text{basis for } E_{\lambda_2}(T) \\
&\quad \vdots && \quad \vdots \\
&\quad v_{r,1}, v_{r,2}, \ldots, v_{r,n_r}] && \leftarrow \text{basis for } E_{\lambda_r}(T)
\end{aligned}$$

Then, β forms a linearly independent set.
[But β might not be a basis for V.]

Moreover, β is a basis for V

iff $n_1 + n_2 + \cdots + n_r = n$

iff T is diagonalizable.