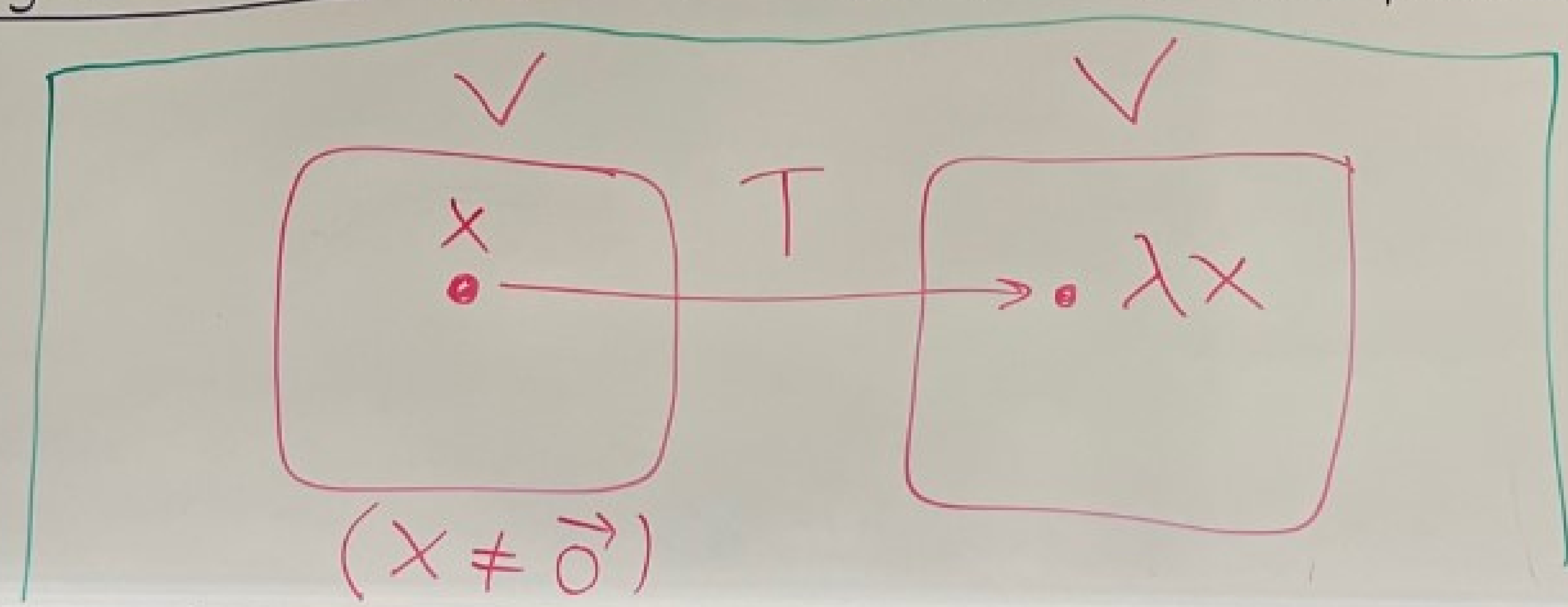
Topic 5 – Eigenvalues, Eigenvectors, and Diagonalization

__Def:__ Let V be a vector space over a field F.
Let $T: V \to V$ be a linear transformation.
If $x \in V$ and $x \neq \vec{0}$ and $T(x) = \lambda x$
where $\lambda \in F$, then we say that x is an
__eigenvector__ of T with corresponding __eigenvalue__ λ.



$\boxed{\lambda = 0 \text{ is okay}}$

$\underline{\text{Ex:}}$ Let $V = \mathbb{R}^2$, $F = \mathbb{R}$.

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a+3b \\ 4a+2b \end{pmatrix}$

[You can check that T is a linear transformation]

Note that

$$T\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1+3(-1) \\ 4(1)+2(-1) \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} = -2\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

So $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector with eigenvalue -2.

on

Also we see that

$$T\begin{pmatrix}3\\4\end{pmatrix}=\begin{pmatrix}3+3(4)\\4(3)+2(4)\end{pmatrix}=\begin{pmatrix}15\\20\end{pmatrix}=5\begin{pmatrix}3\\4\end{pmatrix}$$

So, $\begin{pmatrix}3\\4\end{pmatrix}$ is an eigenvector with eigenvalue 5.

Ex: Let

$$V = P_2(\mathbb{R}) = \{a + bx + cx^2 \mid a, b, c \in \mathbb{R}\}$$

$$F = \mathbb{R}$$

$$T : P_2(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$$

$$T(a + bx + cx^2) = b + 2cx$$

$T(f) = f'$

We showed T is a linear transformation previously

We have that

$$T(1) = 0 = 0 \cdot 1$$

$a=1, b=0, c=0$

So, 1 is an eigenvector with eigenvalue $\lambda = 0$.

$$T(2) = 0 = 0 \cdot 2$$

$a=2, b=0, c=0$

$2 = 2 \cdot 1 + 0x + 0x^2$

eigenvector $= 2$

eigenvalue $= 0$

Def: Let V be a vector space over a field F.
Let $T: V \to V$ be a linear transformation.
We say that T is **diagonalizable** if
there exists an ordered basis β for V
where $[T]_\beta$ is a diagonal matrix.

A **diagonal matrix** is of the form
$$\begin{pmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{pmatrix}, \quad d_i \in F$$

Examples of diagonal matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$, d_i \in F$

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & i \end{pmatrix}$$

Ex: Let $V = \mathbb{R}^2$, $F = \mathbb{R}$.

Let $T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a+3b \\ 4a+2b \end{pmatrix}$

Eigenvectors: $\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

Eigenvalues: $-2, 5$

Let's show that T is diagonalizable.

Let $\beta = \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right]$.

You can check that β is a linearly independent set of 2 vectors and thus is a basis for $V = \mathbb{R}^2$.

Let's make $[T]_\beta = [T]_\beta^\beta$

$$T\begin{pmatrix} 1 \\ -1 \end{pmatrix} = -2\begin{pmatrix} 1 \\ -1 \end{pmatrix} = -2\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 0\begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$T\begin{pmatrix} 3 \\ 4 \end{pmatrix} = 5\begin{pmatrix} 3 \\ 4 \end{pmatrix} = 0\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 5\begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

So,

$$[T]_\beta = \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix} \leftarrow \text{diagonal}$$

So, T is diagonalizable.

What is the point of this?

Let $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

Then, $\beta = [v_1, v_2]$ is our ordered basis for $V = \mathbb{R}^2$.

Let v be some vector in $V = \mathbb{R}^2$.

Then, $v = c_1 v_1 + c_2 v_2$ for unique constants $c_1, c_2 \in F$.

And,

T is linear

$$T(v) = T(c_1 v_1 + c_2 v_2) = c_1 T(v_1) + c_2 T(v_2)$$

$$= c_1(-2v_1) + c_2(5v_2)$$

$$= -2c_1 v_1 + 5c_2 v_2$$

That is,

$$T(c_1 v_1 + c_2 v_2) = -2(c_1 v_1) + 5(c_2 v_2)$$

When T is diagonalizable it has a coordinate system that makes T's formula as simple as possible

Each term is scaled by an eigenvalue.

In matrix notation:

$$T(c_1v_1+c_2v_2)=-2(c_1v_1)+5(c_2v_2)$$

is the same as

$$\underbrace{\begin{pmatrix}-2&0\\0&5\end{pmatrix}}_{[T]_\beta}\underbrace{\begin{pmatrix}c_1\\c_2\end{pmatrix}}_{[v]_\beta}=\underbrace{\begin{pmatrix}-2c_1\\5c_2\end{pmatrix}}_{[T(v)]_\beta}$$

<u>Theorem:</u> Let V be a finite-dimensional vector space over a field F.

Let $T: V \to V$ be a linear transformation.

T is diagonalizable if and only if

there exists an ordered basis $\beta = [v_1, v_2, \dots, v_n]$

of V consisting of eigenvectors of T.

Moreover, if this is the case then

$$[T]_\beta = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

where λ_i is the eigenvalue cooresponding to v_i

proof:

T is diagonalizable

iff there exists an ordered basis $\beta = [v_1, v_2, \cdots, v_n]$ for V such that

$$[T]_\beta = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

that is, $[T]_\beta$ is diagonal

Where $\lambda_1, \lambda_2, \cdots, \lambda_n \in F$

$\boxed{\text{iff}}$ there exists an ordered basis $\beta = \{v_1, v_2, \ldots, v_n\}$ of V where

$$T(v_1) = \lambda_1 v_1 + 0v_2 + 0v_3 + \cdots + 0v_n = \lambda_1 v_1$$

$$T(v_2) = 0v_1 + \lambda_2 v_2 + 0v_3 + \cdots + 0v_n = \lambda_2 v_2$$

$$T(v_3) = 0v_1 + 0v_2 + \lambda_3 v_3 + \cdots + 0v_n = \lambda_3 v_3$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$T(v_n) = 0v_1 + 0v_2 + 0v_3 + \cdots + \lambda_n v_n = \lambda_n v_n$$

$\boxed{\text{iff}}$ there exists an ordered basis $\beta = \{v_1, v_2, \dots, v_n\}$ where

$$T(v_i) = \lambda_i v_i \quad \text{for } 1 \leq i \leq n$$

That is, β is a basis of eigenvectors for T.

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