$\boxed{\text{HW } 3}$

$\boxed{2(c)}$

$$\boxed{\begin{gathered} T: M_{2,3}(\mathbb{R}) \longrightarrow M_{2,2}(\mathbb{R}) \\ T\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = \begin{pmatrix} 2a-b & c+2d \\ 0 & 0 \end{pmatrix} \end{gathered}}$$

$\boxed{T \text{ is linear}}$:

Let $v_1, v_2 \in M_{2,3}(\mathbb{R})$ and $\alpha, \beta \in \mathbb{R}$.

Then, $v_1 = \begin{pmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \end{pmatrix}$.

Then, $T(\alpha v_1 + \beta v_2) = T\left(\alpha \begin{pmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \end{pmatrix} + \beta \begin{pmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \end{pmatrix}\right) = T\begin{pmatrix} \alpha a_1 + \beta a_2 & \alpha b_1 + \beta b_2 & \alpha c_1 + \beta c_2 \\ \alpha d_1 + \beta d_2 & \alpha e_1 + \beta e_2 & \alpha f_1 + \beta f_2 \end{pmatrix}$

$= \Big( 2($

$= \Big( 2\alpha a$

$= \alpha \Big( 2a$

$= \alpha T(v_1$

$+\beta c_2$
$+\beta f_2$

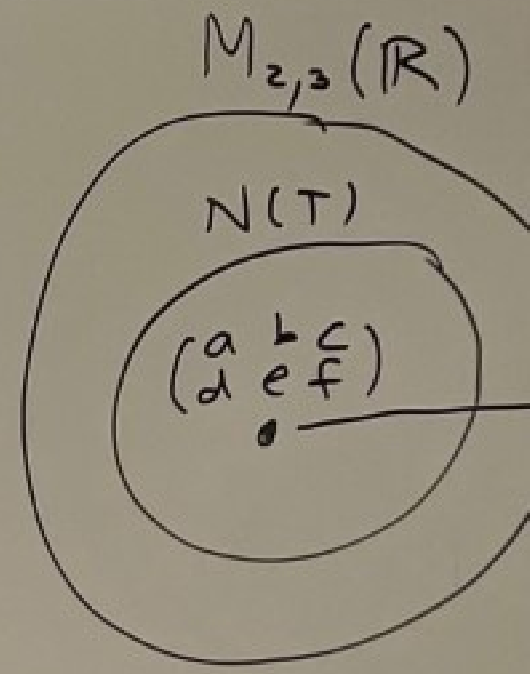
$$\begin{pmatrix} 2(\alpha a_1+\beta a_2)-(\alpha b_1+\beta b_2) & \alpha c_1+\beta c_2+2(\alpha d_1+\beta d_2) \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2\alpha a_1-\alpha b_1 & \alpha c_1+2\alpha d_1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 2\beta a_2-\beta b_2 & \beta c_2+2\beta d_2 \\ 0 & 0 \end{pmatrix}$$

$$= \alpha \begin{pmatrix} 2a_1-b_1 & c_1+2d_1 \\ 0 & 0 \end{pmatrix} + \beta \begin{pmatrix} 2a_2-b_2 & c_2+2d_2 \\ 0 & 0 \end{pmatrix}$$

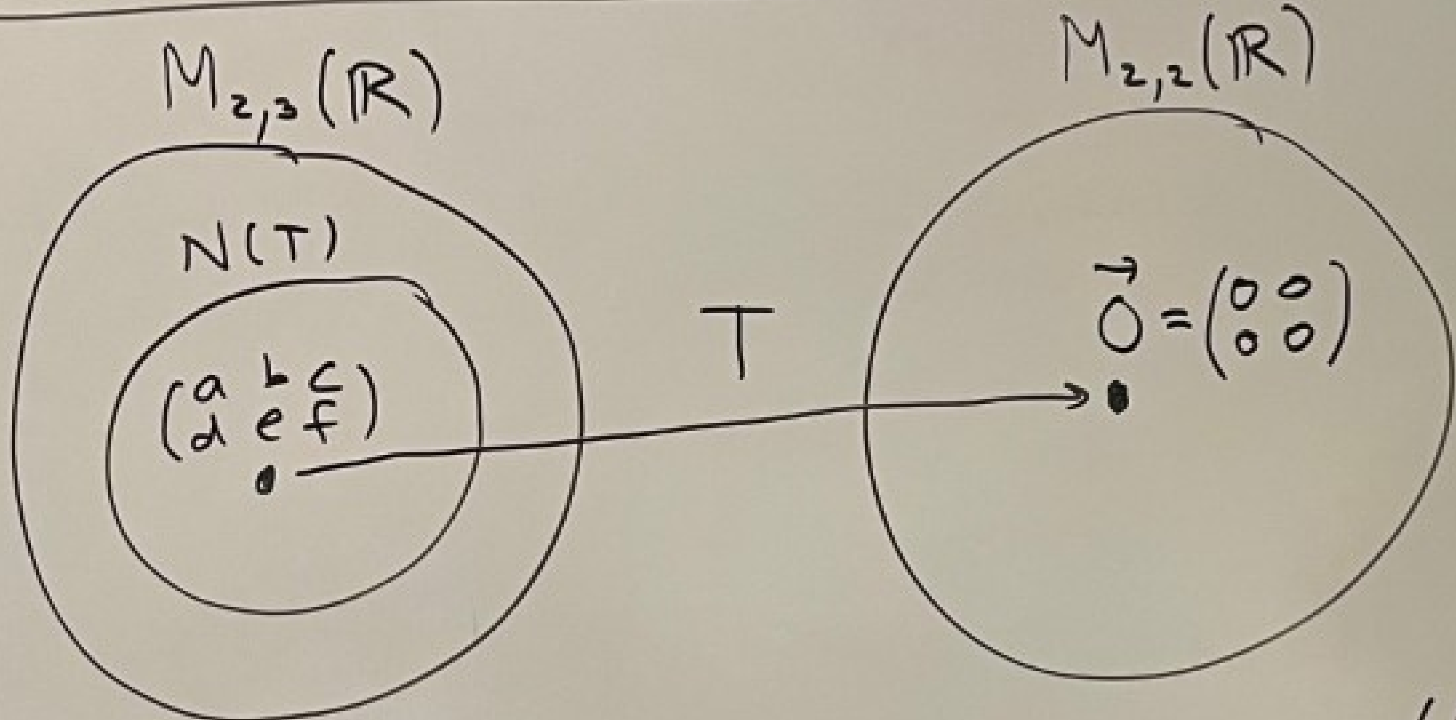
$$= \alpha T(v_1) + \beta T(v_2) \quad \blacksquare$$

What is N(
Find a basis




$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \in N(T$

What is $N(T)$?
Find a basis for $N(T)$




$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \in N(T)$ iff $T\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

iff $\begin{pmatrix} 2a-b & c+2d \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ iff

$$\boxed{\begin{aligned} 2a-b &= 0 \\ c+2d &= 0 \end{aligned}}$$

$$\begin{aligned} 2a - b &= 0 \\ c + 2d &= 0 \end{aligned} \quad \xrightarrow{\frac{1}{2}R_1 \to R_1} \quad \begin{aligned} a - \tfrac{1}{2}b &= 0 \\ c + 2d &= 0 \end{aligned}$$

leading variables: $a, c$

free variables: $b, d, e, f$

Solve for leading.
Give free var. new names

$$\Rightarrow \quad \begin{aligned} a &= \tfrac{1}{2}b & (1) \\ c &= -2d & (2) \\ b &= s & (3) \\ d &= t & (4) \\ e &= u & (5) \\ f &= v & (6) \end{aligned}$$

$\Rightarrow$

back-substitution

(6) $f = v$
(5) $e = u$
(4) $d = t$
(3) $b = s$
(2) $c = -2t$
(1) $a = \frac{1}{2}s$

$s, t, u, v$ are any real numbers

$\Rightarrow$

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \in N(T)$$

iff

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = \begin{pmatrix} \frac{1}{2}s & s & -2t \\ t & u & v \end{pmatrix}$$

where $s, t, u, v \in \mathbb{R}$

## Basis for N(T)

$$\begin{pmatrix} \frac{1}{2}s & s & -2t \\ t & u & v \end{pmatrix} = s\begin{pmatrix} \frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + t\begin{pmatrix} 0 & 0 & -2 \\ 1 & 0 & 0 \end{pmatrix} + u\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + v\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\boxed{T\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = \begin{pmatrix} 2a-b & c+2d \\ 0 & 0 \end{pmatrix}}$$

Note that: $\begin{pmatrix} \frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -2 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ are in $N(T)$.

Thus,

$$N(T) = \text{span}\left(\left\{\begin{pmatrix} \frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -2 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}\right)$$

Note there vectors are lin. ind. because

$$c_1\begin{pmatrix} \frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + c_2\begin{pmatrix} 0 & 0 & -2 \\ 1 & 0 & 0 \end{pmatrix} + c_3\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + c_4\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\vec{0}}$$

iff

$$\begin{pmatrix} \frac{1}{2}c_1 & c_1 & -2c_2 \\ c_2 & c_3 & c_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

iff

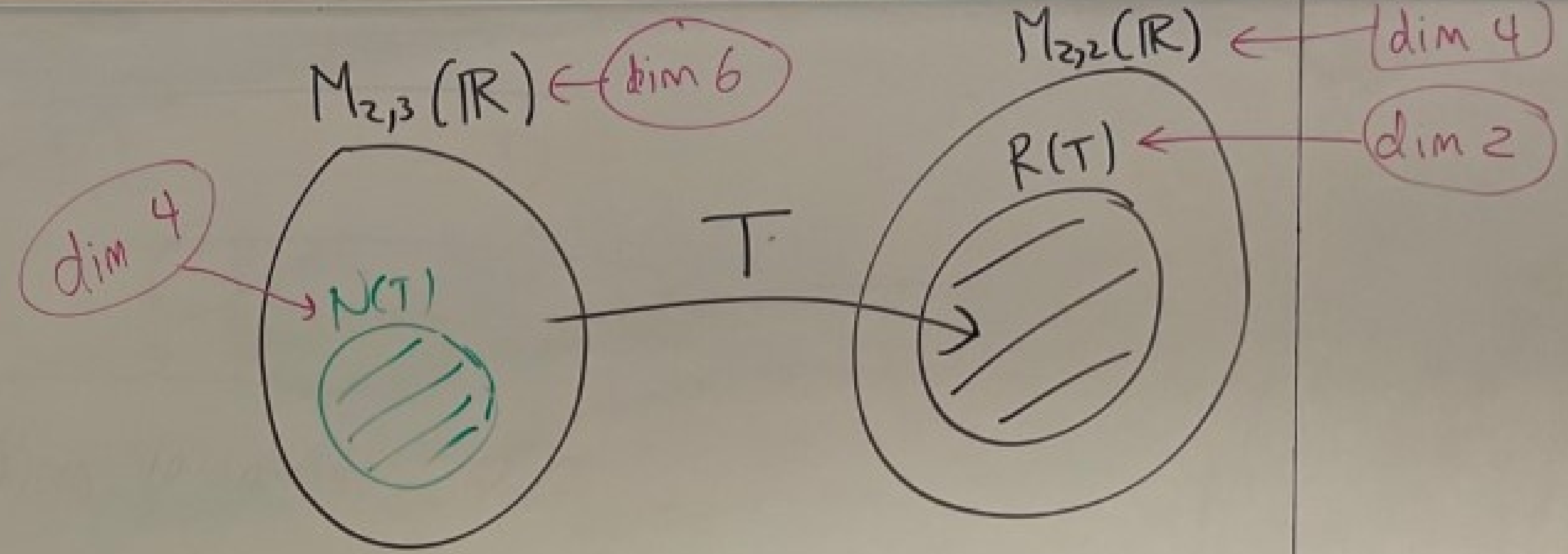
$$c_1 = 0,\quad c_2 = 0,\quad c_3 = 0,\quad c_4 = 0.$$

Thus, $\begin{pmatrix} \frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -2 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is a basis for $N(T)$.

So, $\text{nullity}(T) = \dim(N(T)) = 4$

What is rank(T)?

$$\text{rank}(T) = \dim(\underbrace{R(T)}_{\text{range of } T})$$




We can use the rank-nullity thm.

$$\dim(M_{2,3}(\mathbb{R})) = \dim(N(T)) + \dim(R(T))$$

$$6 = 4 + \dim(R(T))$$

$$\dim(R(T)) = 2.$$

(dim 4)

(dim 2)

Is T one-to-one?

T is one-to-one iff $\underbrace{\dim(N(T)) = 0}_{\text{iff } N(T) = \{\vec{0}\}}$

Since $\dim(N(T)) = 2$, this means T is not one-to-one.

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Is T onto?

T is onto iff $R(T) = M_{2,2}(\mathbb{R})$

This isn't the case since $\dim(R(T)) = 2$ and $\dim(M_{2,2}(\mathbb{R})) = 4$.

$= 0$

$\{\vec{0}\}$

4.

$$\boxed{T\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = \begin{pmatrix} 2a-b & c+2d \\ 0 & 0 \end{pmatrix}}$$

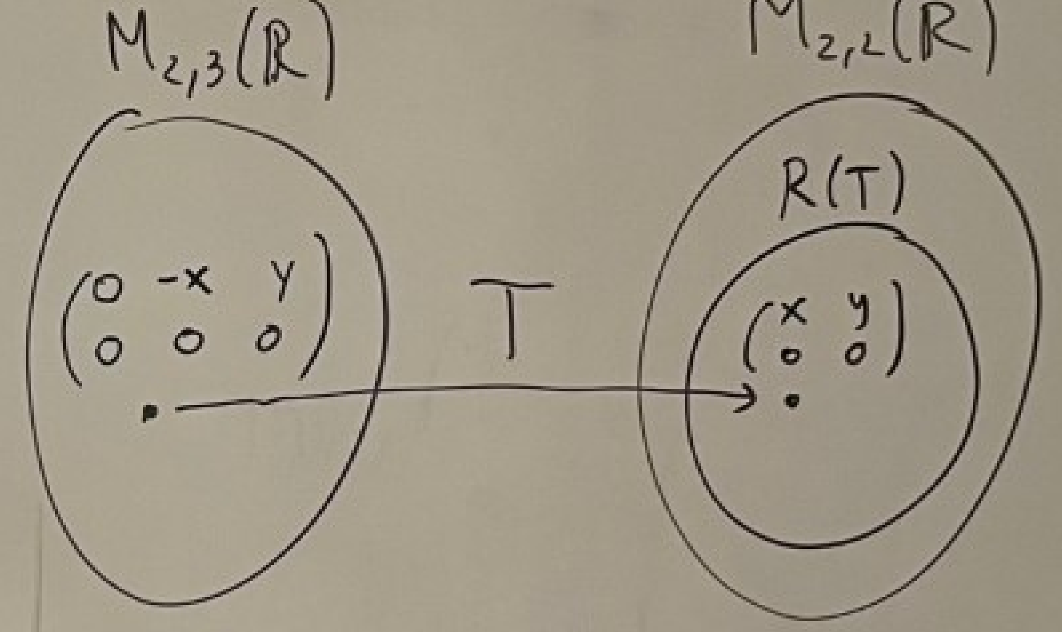
## What is $R(T)$?

**Claim:** $R(T) = \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \;\middle|\; x, y \in \mathbb{R} \right\}$

Given $\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$

we see

$$T\begin{pmatrix} 0 & -x & y \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$$



$\boxed{\text{HW 4}}$ ① $T:\mathbb{R}^3 \to \mathbb{R}^3$, $T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a+b \\ c \\ -a \end{pmatrix}$

$$\beta' = \left[ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right]$$

Find $[T]_{\beta'}$.

Recall: $[T]_{\beta'} = [T]_{\beta'}^{\beta'}$

$$\underbrace{T\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}}_{\text{plug in } \beta'} \underbrace{= a\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + b\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\text{write in terms of } \beta'}$$

$$T\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = d\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + e\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + f\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = g\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + h\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + i\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Need to solve

$$\begin{pmatrix}1\\1\\-1\end{pmatrix}=\begin{pmatrix}a+b\\2b\\a+b+c\end{pmatrix}$$

$$\begin{pmatrix}1\\2\\1\end{pmatrix}=\begin{pmatrix}d+e\\2e\\d+e+f\end{pmatrix}$$

$$\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}g+h\\2h\\g+h+i\end{pmatrix}$$

$\Rightarrow$

$$\begin{aligned}a+b&=1\\2b&=1\\a+b+c&=-1\end{aligned}$$

$$\begin{aligned}d+e&=1\\2e&=2\\d+e+f&=1\end{aligned}$$

$$\begin{aligned}g+h&=0\\2h&=0\\g+h+i&=1\end{aligned}$$

$\Rightarrow$

$a=1/2,\ b=1/2,\ c=-2$

$d=5/2,\ e=1/2,\ f=-4$

$g=-1/2,\ h=1/2,\ i=0$

$\Downarrow$

$$[T]_{\beta'}=\begin{pmatrix}a&d&g\\b&e&h\\c&f&i\end{pmatrix}=\begin{pmatrix}1/2&5/2&-1/2\\1/2&1/2&1/2\\-2&-4&0\end{pmatrix}$$

What if you knew $[x]_{\beta'}=(2,4,-2)$

What is $x$? What is $[T(x)]_{\beta'}$?

$$[x]_{\beta'} = (2, 4, -2)$$

$$\beta' = \left[ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right]$$

$$x = 2 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 4 \cdot \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} - 2 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \\ 4 \end{pmatrix}$$

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$$[T(x)]_{\beta'} = [T]_{\beta'}[x]_{\beta'} = \begin{pmatrix} 1/2 & 5/2 & -1/2 \\ 1/2 & 1/2 & 1/2 \\ -2 & -4 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 + 10 + 1 \\ 1 + 2 - 1 \\ -4 - 16 + 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 12 \\ 2 \\ -20 \end{pmatrix}$$