$\boxed{\text{Last time}}$

$$[T]_\beta = Q^{-1}\,[T]_{\beta'}\,Q$$

$$[I]_{\beta'}^{\beta} \quad [T]_{\beta'} \quad [I]_{\beta}^{\beta'}$$

<u>Def:</u> Let A and B be $n\times n$ matrices with entries from a field F. We say that A and B are <u>similar</u> if there exists an $n\times n$ invertible matrix Q with entries from F where $B = Q^{-1}AQ$.

So last time's Corollary says $[T]_\beta$ and $[T]_{\beta'}$ are similar matrices.

<u>Theorem:</u> Let V be a finite-dimensional vector space over a field F. Let β be an ordered basis for V. Let $T: V \to V$ be a linear transformation. Suppose $n = \dim(V)$.

If A is an $n \times n$ matrix with entries from F that is similar to $[T]_\beta$, then there exists an ordered basis β' for V where $A = [T]_{\beta'}$.

<u>proof:</u> See notes on website $\square$

Topic 4.5 – Review of determinants

<u>Def:</u> Let A be an $n \times n$ matrix with entries from a field F. Let $1 \leq i \leq n$ and $1 \leq j \leq n$. The matrix A_{ij} is defined to be the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j from A.

Ex:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

A is 3×3

$$A_{12} = \begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix}$$

$$A = \begin{pmatrix} \cancel{1} & \cancel{2} & \cancel{3} \\ 4 & \cancel{5} & 6 \\ 7 & \cancel{8} & 9 \end{pmatrix}$$

$$A_{23} = \begin{pmatrix} 1 & 2 \\ 7 & 8 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 2 & \cancel{3} \\ \cancel{4} & \cancel{5} & \cancel{6} \\ 7 & 8 & \cancel{9} \end{pmatrix}$$

Def: Let A be an $n \times n$ matrix with entries from a field F.
Let a_{ij} be the entry in A at row i and column j.

(1) If $n=1$ and $A = (a_{11})$, then define $\det(A) = a_{11}$

(2) If $n=2$ and $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ then define $\det(A) = a_{11}a_{22} - a_{12}a_{21}$

(3) If $n \geq 3$, then calculate the determinant as follows.
Pick a column j to "expand on" where $1 \leq j \leq n$.
Define

$$\det(A) = \underbrace{\sum_{i=1}^{n}}_{\substack{\text{sum over rows } i \\ \text{column } j \text{ is fixed}}} (-1)^{i+j} a_{ij} \det(A_{ij})$$

Note: In 3 you can instead pick a row i to "expand on" where $1 \leq i \leq n$.

And define

$$\det(A) = \underbrace{\sum_{j=1}^{n}}_{\substack{\text{sum over columns } j \\ \text{keep row } i \text{ fixed}}} (-1)^{i+j} a_{ij} \det(A_{ij})$$

One can show that it doesn't matter what column or row you pick at each step of the determinant calculation. The end result will be the same.

Notation:

We use $|\cdot|$ for determinant sometimes.

For example,

$$\det\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix} = \begin{vmatrix}1&2&3\\4&5&6\\7&8&9\end{vmatrix}$$

Ex: $\det(-5) = -5$

Ex: $\det\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = (1)(4) - (2)(3) = 4 - 6 = -2$

Ex: Let $A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 2 & -2 \\ 1 & -1 & 0 \end{pmatrix}$

Calculate $\det(A)$

Let's expand on column $j=2$.

$$\det\begin{pmatrix} 1 & 0 & -1 \\ 2 & 2 & -2 \\ 1 & -1 & 0 \end{pmatrix} = \underbrace{(-1)^{1+2} a_{12} \det(A_{12})}_{j=2,\ i=1} + \underbrace{(-1)^{2+2} a_{22} \det(A_{22})}_{j=2,\ i=2} + \underbrace{(-1)^{3+2} a_{32} \det(A_{32})}_{j=2,\ i=3}$$

$$= (-1)\cdot 0 \cdot \underbrace{\begin{vmatrix} 2 & -2 \\ 1 & 0 \end{vmatrix}} + (1)\cdot 2 \cdot \underbrace{\begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix}} + (-1)\cdot \underbrace{(-1)\cdot \begin{vmatrix} 1 & -1 \\ 2 & -2 \end{vmatrix}}$$

$$\begin{pmatrix} (-1)^{1+1} & (-1)^{1+2} & (-1)^{1+3} \\ (-1)^{2+1} & (-1)^{2+2} & (-1)^{2+3} \\ (-1)^{3+1} & (-1)^{3+2} & (-1)^{3+3} \end{pmatrix} = \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & -1 \\ 2 & 2 & -2 \\ 1 & -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & -1 \\ 2 & 2 & -2 \\ 1 & -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & -1 \\ 2 & 2 & -2 \\ 1 & -1 & 0 \end{pmatrix}$$

$$= 0 + 2\left[(1)(0) - (-1)(1)\right] + \left[(1)(-2) - (-1)(2)\right] = 2 + 0 = \boxed{2}$$

Ex:

$$\det\begin{pmatrix} 2 & 0 & 0 & 0 \\ 3 & 1 & 0 & -1 \\ 5 & 2 & 2 & -2 \\ 10 & 1 & -1 & 0 \end{pmatrix} = (+1)(2)\begin{vmatrix} 1 & 0 & -1 \\ 2 & 2 & -2 \\ 1 & -1 & 0 \end{vmatrix} - (0)\cdot\begin{vmatrix} 3 & 0 & -1 \\ 5 & 2 & -2 \\ 10 & -1 & 0 \end{vmatrix}$$

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 3 & 1 & 0 & -1 \\ 5 & 2 & 2 & -2 \\ 10 & 1 & -1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 2 & 0 & 0 & 0 \\ 3 & 1 & 0 & -1 \\ 5 & 2 & 2 & -2 \\ 10 & 1 & -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix}$$

$$+(0)\begin{vmatrix} 3 & 1 & -1 \\ 5 & 2 & -2 \\ 10 & 1 & 0 \end{vmatrix} - (0)\cdot\begin{vmatrix} 3 & 1 & 0 \\ 5 & 2 & 2 \\ 10 & 1 & -1 \end{vmatrix}$$

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 3 & 1 & 0 & -1 \\ 5 & 2 & 2 & -2 \\ 10 & 1 & -1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 2 & 0 & 0 & 0 \\ 3 & 1 & 0 & -1 \\ 5 & 2 & 2 & -2 \\ 10 & 1 & -1 & 0 \end{pmatrix}$$

$$2\begin{vmatrix} 1 & 0 & -1 \\ 2 & 2 & -2 \\ 1 & -1 & 0 \end{vmatrix} + 0 + 0 + 0 = 2(2) = 4$$

got 2 for this earlier

Theorem Let A and B be $n \times n$ matrices with entries from a field F.

Then:

1. $\det(AB) = \det(A) \cdot \det(B)$
2. A is invertible iff $\det(A) \neq 0$

   If A is invertible then $\det(A^{-1}) = [\det(A)]^{-1}$