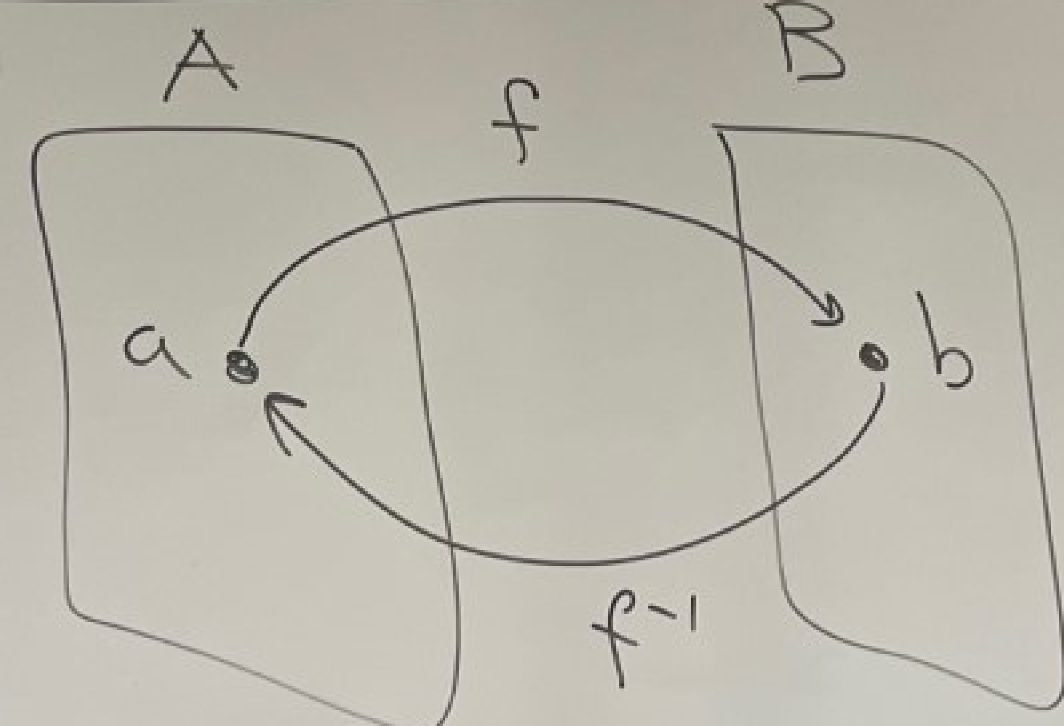
__Recall:__ Let $f : A \to B$ be a one-to-one and onto function

We define $f^{-1} : B \to A$ by

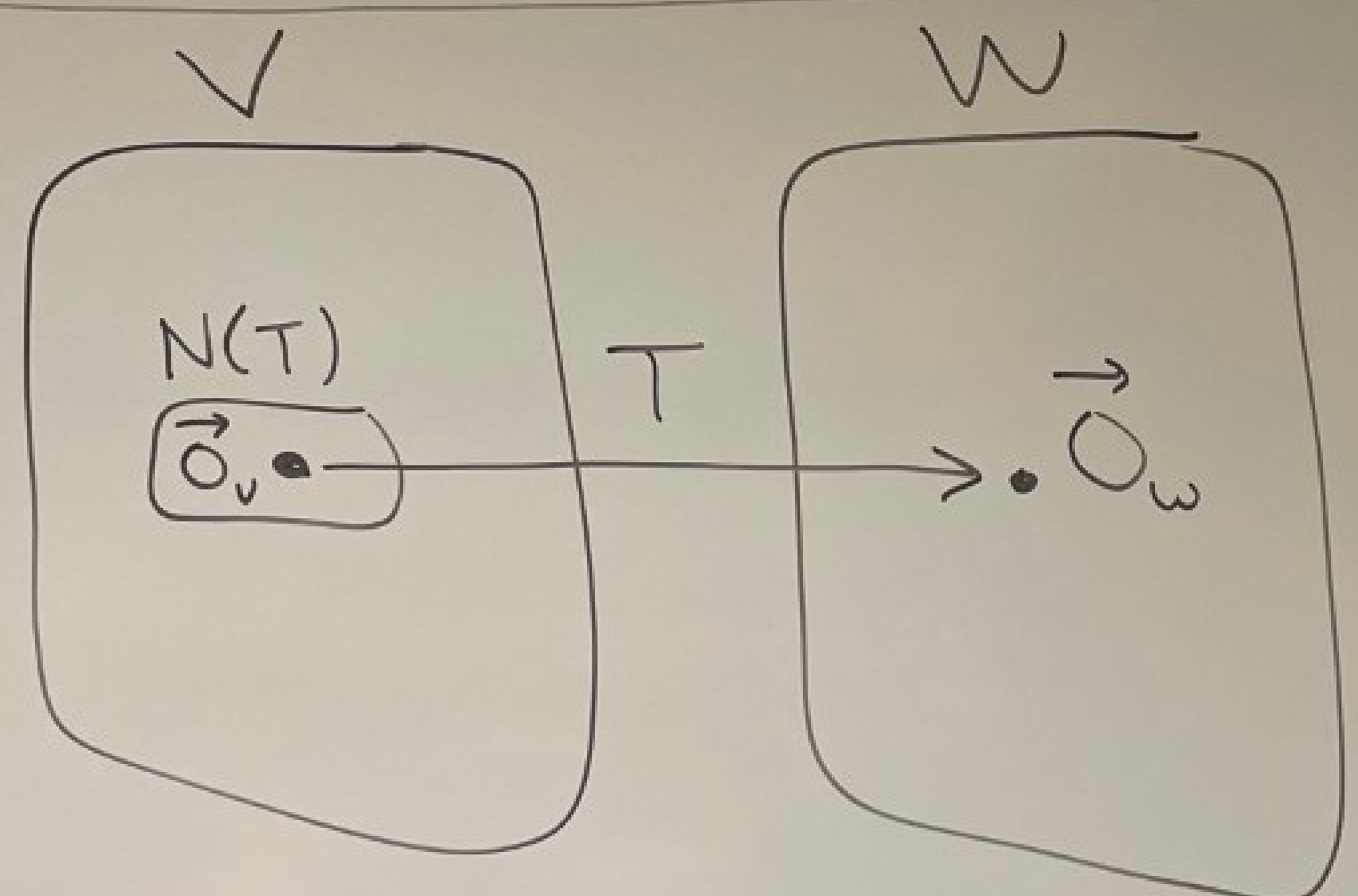
$$f^{-1}(b) = a \text{ iff } f(a) = b$$



HW 3 – #6(a) Let $T: V \to W$ be a linear transformation.

Then: T is one-to-one iff $N(T) = \{\vec{0}_V\}$

iff $\dim(N(T)) = 0$



E

be

Claim:

proof: Let x

Then, $T(\alpha x$

... be

$) = \{\vec{0}_V\}$

$T)) = 0$

<u>Ex:</u> Let $T : \mathbb{R}^2 \to \mathbb{R}^2$

be defined by

$$T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a+b \\ a-b \end{pmatrix}$$

$T : V \to W$

$V = \mathbb{R}^2$

$W = \mathbb{R}^2$

$F = \mathbb{R}$

Claim: T is a linear transformation

<u>proof:</u> Let $x, y \in \mathbb{R}^2$ and $\alpha, \beta \in \mathbb{R}$. Then, $x = \begin{pmatrix} a \\ b \end{pmatrix}$, $y = \begin{pmatrix} c \\ d \end{pmatrix}$ where $a, b, c, d \in \mathbb{R}$.

Then, $T(\alpha x + \beta y) = T\left(\alpha\begin{pmatrix} a \\ b \end{pmatrix} + \beta\begin{pmatrix} c \\ d \end{pmatrix}\right) = T\left(\begin{pmatrix} \alpha a \\ \alpha b \end{pmatrix} + \begin{pmatrix} \beta c \\ \beta d \end{pmatrix}\right)$

$$= T\begin{pmatrix} \alpha a + \beta c \\ \alpha b + \beta d \end{pmatrix} = \begin{pmatrix} \alpha a + \beta c + \alpha b + \beta d \\ \alpha a + \beta c - \alpha b - \beta d \end{pmatrix} = \begin{pmatrix} \alpha a + \alpha b \\ \alpha a - \alpha b \end{pmatrix} + \begin{pmatrix} \beta c + \beta d \\ \beta c - \beta d \end{pmatrix}$$

$$= \alpha\begin{pmatrix} a+b \\ a-b \end{pmatrix} + \beta\begin{pmatrix} c+d \\ c-d \end{pmatrix} = \alpha T(x) + \beta T(y) \quad \blacksquare$$

$\boxed{\text{Q: Is } T \text{ one-to-one?}}$

Let's calculate $N(T)$.

$\begin{pmatrix} a \\ b \end{pmatrix} \in N(T)$

iff $T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

iff $\boxed{\begin{matrix} a+b=0 \\ a-b=0 \end{matrix}}$ $\boxed{\text{iff}}$ $\boxed{\begin{matrix} a+b=0 \\ b=0 \end{matrix}}$ iff $\boxed{\begin{matrix} a=0 \\ b=0 \end{matrix}}$ iff $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right) \xrightarrow{-R_1+R_2\to R_2} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -2 & 0 \end{array}\right) \xrightarrow{-\frac{1}{2}R_2\to R_2} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ \underbrace{0 \quad 1}_{\text{row echelon form}} & & 0 \end{array}\right)$$

So, $N(T) = \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}$

By HW 3-6(a)

T is one-to-one.

$\boxed{Q}$

So, $N(T) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$

So, $\underbrace{\dim(N(T))}_{\text{nullity of } T} = 0$

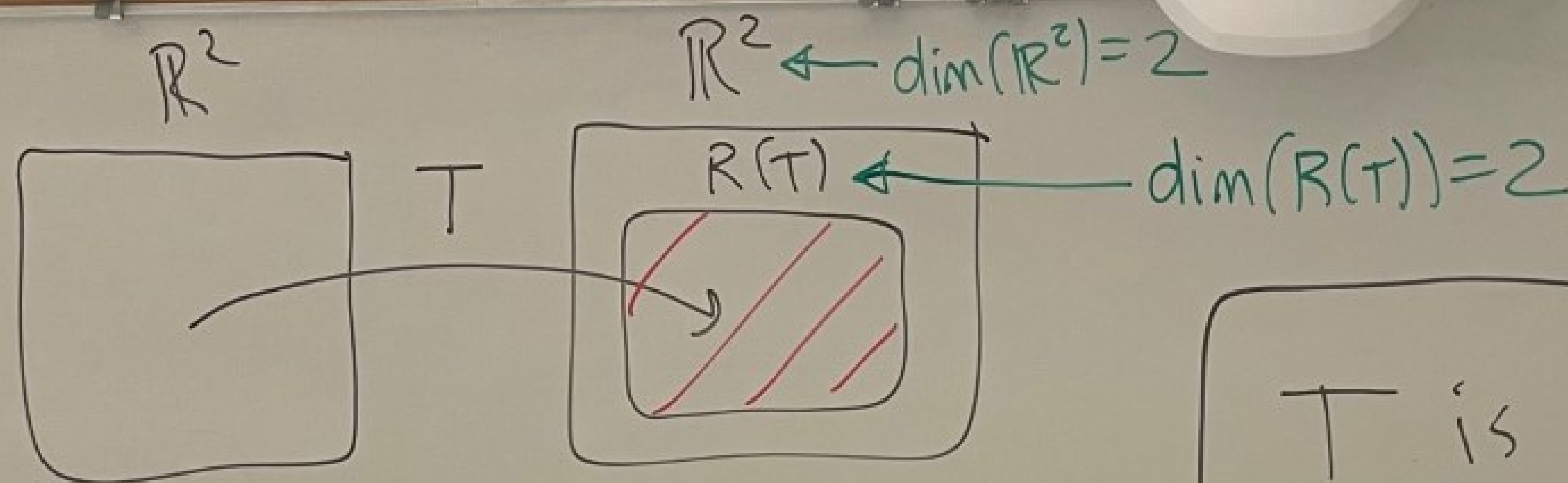
$\boxed{\text{Q: Is } T \text{ onto?}}$

By the rank-nullity theorem

$$\dim(\mathbb{R}^2) = \dim(N(T)) + \dim(R(T))$$

$$2 = 0 + \dim(R(T))$$

So, $\underbrace{\dim(R(T))}_{\text{rank of } T} = 2$



By a theorem in class, since $R(T)$ is a subspace of $\mathbb{R}^2$ and $\dim(R(T)) = 2 = \dim(\mathbb{R}^2)$ we must have $R(T) = \mathbb{R}^2$.

So, $\boxed{T \text{ is onto } \mathbb{R}^2}$

T is one-to-one and onto.

So, let's calculate T^{-1}!

$$T^{-1}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

iff $T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$

iff $\begin{pmatrix} a+b \\ a-b \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$

iff $\boxed{\begin{aligned} a+b &= c \\ a-b &= d \end{aligned}}$

$$\left(\begin{array}{cc|c} 1 & 1 & c \\ \boxed{1} & -1 & d \end{array}\right) \xrightarrow{-R_1+R_2 \to R_2} \left(\begin{array}{cc|c} 1 & 1 & c \\ 0 & -2 & d-c \end{array}\right)$$

make 0

$$\xrightarrow{-\frac{1}{2}R_2 \to R_2} \left(\begin{array}{cc|c} 1 & 1 & c \\ 0 & 1 & -\frac{d}{2}+\frac{c}{2} \end{array}\right)$$

row echelon form

$$\boxed{\begin{aligned} a+b &= c \quad (1) \\ b &= -\frac{d}{2}+\frac{c}{2} \quad (2) \end{aligned}}$$

$\Rightarrow$

(2) $b = -\frac{d}{2}+\frac{c}{2}$

(1) $a = c-b = c-\left(-\frac{d}{2}+\frac{c}{2}\right) = \frac{c}{2}+\frac{d}{2}$

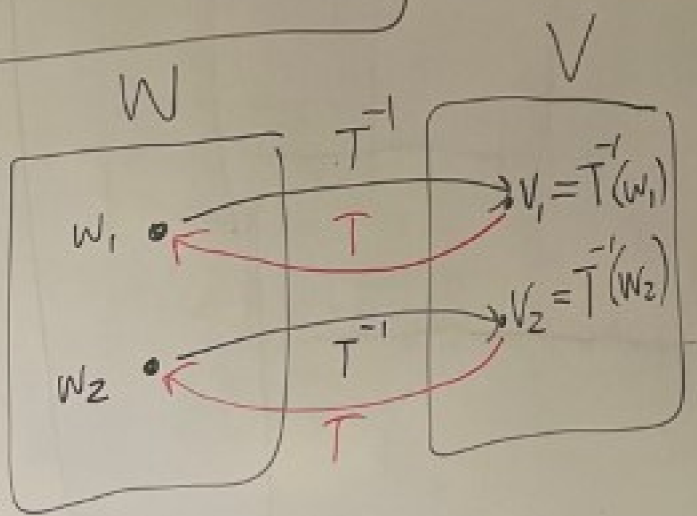
So, $T^{-1}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} \frac{c}{2}+\frac{d}{2} \\ \frac{c}{2}-\frac{d}{2} \end{pmatrix} \begin{matrix} \leftarrow a \\ \leftarrow b \end{matrix}$

You can check that T^{-1} is also a linear transformation

Theorem: Let V and W are vector spaces over a field F. Let $T: V \to W$ be a linear transformation that is one-to-one and onto. Then, $T^{-1}: W \to V$ is a linear transformation.

proof:

Let $w_1, w_2 \in W$ and $\alpha_1, \alpha_2 \in F$. We want to show $T^{-1}(\alpha_1 w_1 + \alpha_2 w_2) = \alpha_1 T^{-1}(w_1) + \alpha_2 T^{-1}(w_2)$



Let $v_1 = T^{-1}(w_1)$ and $v_2 = T^{-1}(w_2)$

Then, $T(v_1) = w_1$ and $T(v_2) = w_2$.

So,

$$\begin{aligned}
T^{-1}(\alpha_1 w_1 + \alpha_2 w_2) &= T^{-1}(\alpha_1 T(v_1) + \alpha_2 T(v_2)) \\
&= T^{-1}(T(\alpha_1 v_1 + \alpha_2 v_2)) \quad \leftarrow T \text{ is linear} \\
&= \alpha_1 v_1 + \alpha_2 v_2 \quad \leftarrow (T^{-1} \circ T)(x) = x \\
&= \alpha_1 T^{-1}(w_1) + \alpha_2 T^{-1}(w_2)
\end{aligned}$$

$\blacksquare$