Math 4550
9/3/25

For the general dihedral group D_{2n} with $n \geq 3$ we have:

$$D_{2n} = \{1, r, r^2, \ldots, r^{n-1}, s, sr, sr^2, \ldots, sr^{n-1}\}$$

where

$$r^n = 1$$

$$s^2 = 1$$

$$r^{-k} = r^{n-k}$$

$$r^k s = sr^{-k} = sr^{n-k}$$

For a derivation of these see the Judson textbook section 5.2

<u>Note:</u> D_{2n} is non-abelian because

$$rs = sr^{n-1} \neq sr$$

Ex: $(n=4)$

$$D_8 = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$$

where $r^4 = 1$, $s^2 = 1$,

$$r^k s = s r^{-k} = s r^{4-k}$$

$r^4 = 1$

Calculations:

$$r^5 s r^3 = s r^{-5} r^3 = s r^{-2} = s r^4 r^{-2}$$

$\boxed{r^{-2} = r^{-1} r^{-1}}$

$$= s r^2$$

$$s(sr^2)(sr) = r^2 s r = s r^{-2} r$$

$s^2 = 1$

$$= s r^{-1}$$

$$= s r^{4-1} = s r^3$$

$\boxed{r^4 = 1}$

Let's review some Math 2550

Given $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$

Then:

$$AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

$$= \begin{pmatrix} (a \ b) \cdot \begin{pmatrix} e \\ g \end{pmatrix} & (a \ b) \cdot \begin{pmatrix} f \\ h \end{pmatrix} \\ (c \ d) \cdot \begin{pmatrix} e \\ g \end{pmatrix} & (c \ d) \cdot \begin{pmatrix} f \\ h \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}$$

Ex: $\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ 4 & -3 \end{pmatrix}$

$$= \begin{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \end{pmatrix} & \begin{pmatrix} 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ -3 \end{pmatrix} \\ \begin{pmatrix} 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \end{pmatrix} & \begin{pmatrix} 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ -3 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 1 \cdot 3 - 1 \cdot 4 & 1 \cdot (-2) - 1 \cdot (-3) \\ 0 \cdot 3 + 2 \cdot 4 & 0 \cdot (-2) + 2(-3) \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 1 \\ 8 & -6 \end{pmatrix}$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \leftarrow \text{identity matrix}$$

$$IA = AI = A \quad \text{for any } 2 \times 2 \ A$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

A^{-1} exists iff $\underbrace{\det(A) \neq 0}_{ad-bc \neq 0}$

If A^{-1} exists then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Ex: $A = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$

$$\det(A) = 1 \cdot 3 - (-1) \cdot 2 = 5$$

Since $\det(A) \neq 0$, A^{-1} exists

$$A^{-1} = \frac{1}{5}\begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 3/5 & 1/5 \\ -2/5 & 1/5 \end{pmatrix}$$

For any A, B we know:

$$\det(AB) = \det(A) \cdot \det(B)$$

2550 Review done

<u>Theorem:</u> Let

$$GL(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; \begin{matrix} a,b,c,d \in \mathbb{R} \\ ad-bc \neq 0 \end{matrix} \right\}$$

be the set of 2×2 invertible matrices with entries from the real numbers.

It's called the <u>general linear group</u>.

Then $GL(2,\mathbb{R})$ is a group under multiplication.

The identity is $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

The inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

proof: see online notes $\square$

$GL(2,\mathbb{R})$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ • $\begin{pmatrix} \pi & 10 \\ 0 & e \end{pmatrix}$ •

$\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$ • $\begin{pmatrix} 1/2 & 0 \\ 0 & 5 \end{pmatrix}$ •

infinitely many more

Note:

$$\underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}}_{B} = \begin{pmatrix} -2 & -1 \\ -1 & -1 \end{pmatrix}$$

not equal

$$\underbrace{\begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}}_{B}\underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}_{A} = \begin{pmatrix} -1 & -1 \\ -1 & -2 \end{pmatrix}$$

Here $AB \neq BA$.

$GL(2,\mathbb{R})$ is not abelian

Examples so far

groups under addition

$\mathbb{R}$

$\mathbb{Q}$

$\mathbb{Z}$

$\mathbb{Z}_n$

groups under multiplication

$\mathbb{R}^* = \mathbb{R} - \{0\}$

U_n

$GL(2, \mathbb{R}) \leftarrow$ not abelian

group under composition

$D_{2n} \leftarrow$ not abelian

Theorem: Let $\langle G, * \rangle$ be a group. Then:

① The identity e is unique.

② For each element $a \in G$ there exists a unique inverse which we will denote by a^{-1}.

③ If $a \in G$, then $(a^{-1})^{-1} = a$

④ If $a, b \in G$, then
$$(a*b)^{-1} = (b^{-1}) * (a^{-1})$$

Proof:

① Suppose $e_1, e_2 \in G$ are both identity elements.

Then,

$$e_1 * a = a * e_1 = a$$

$$e_2 * a = a * e_2 = a$$

for all $a \in G$.

Then,

$$e_1 = e_1 * e_2 = e_2$$

$\boxed{a = a * e_2}$ $\boxed{e_1 * a = a}$

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(2) Let $a \in G$.

Suppose $b_1, b_2 \in G$ are both inverses for a.

Then,

$$a * b_1 = b_1 * a = e$$

$$a * b_2 = b_2 * a = e$$

Thus,

$$a * b_1 = e = a * b_2$$

Apply b_2 to the left of both sides:

$$b_2 * (a * b_1) = b_2 * (a * b_2)$$

By associativity

$$\underbrace{(b_2 * a)}_{e} * b_1 = \underbrace{(b_2 * a)}_{e} * b_2$$

So,

$$e * b_1 = e * b_2.$$

Thus, $b_1 = b_2$

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3 Since

$$(a^{-1}) * a = e \quad \text{and} \quad a * (a^{-1}) = e$$

by definition

$$(a^{-1})^{-1} = a.$$

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4. We have

$$(a*b) * (b^{-1} * a^{-1})$$

$$= ((a*b) * b^{-1}) * a^{-1}$$

$$= (a * (b * b^{-1})) * a^{-1}$$

$$= (a * e) * a^{-1}$$

$$= a * a^{-1}$$

$$= e$$

So, $(a * b)^{-1} = b^{-1} * a^{-1}$ $\blacksquare$

<u>Notation:</u> When dealing with an abstract group $\langle G, * \rangle$ we will make the following conventions.

- We will just write ab instead of $a*b$.
  For example, $aabcab$ means $a*a*b*c*a*b$.
- By associativity we never need to use parenthesis.
- If n is a positive integer, then

$$a^n = \underbrace{aaaa\cdots a}_{n \text{ times}}$$

$$a^{-n} = \underbrace{a^{-1}a^{-1}a^{-1}\cdots a^{-1}}_{n \text{ times}}$$

$$a^0 = e$$

For example: $a^3 = aaa$

$a^{-4} = a^{-1}a^{-1}a^{-1}a^{-1}$

- We will also just say "Let G be a group" and not write $*$.
- In HW 1 I kept the $*$ notation but after HW 1 we won't use it anymore.