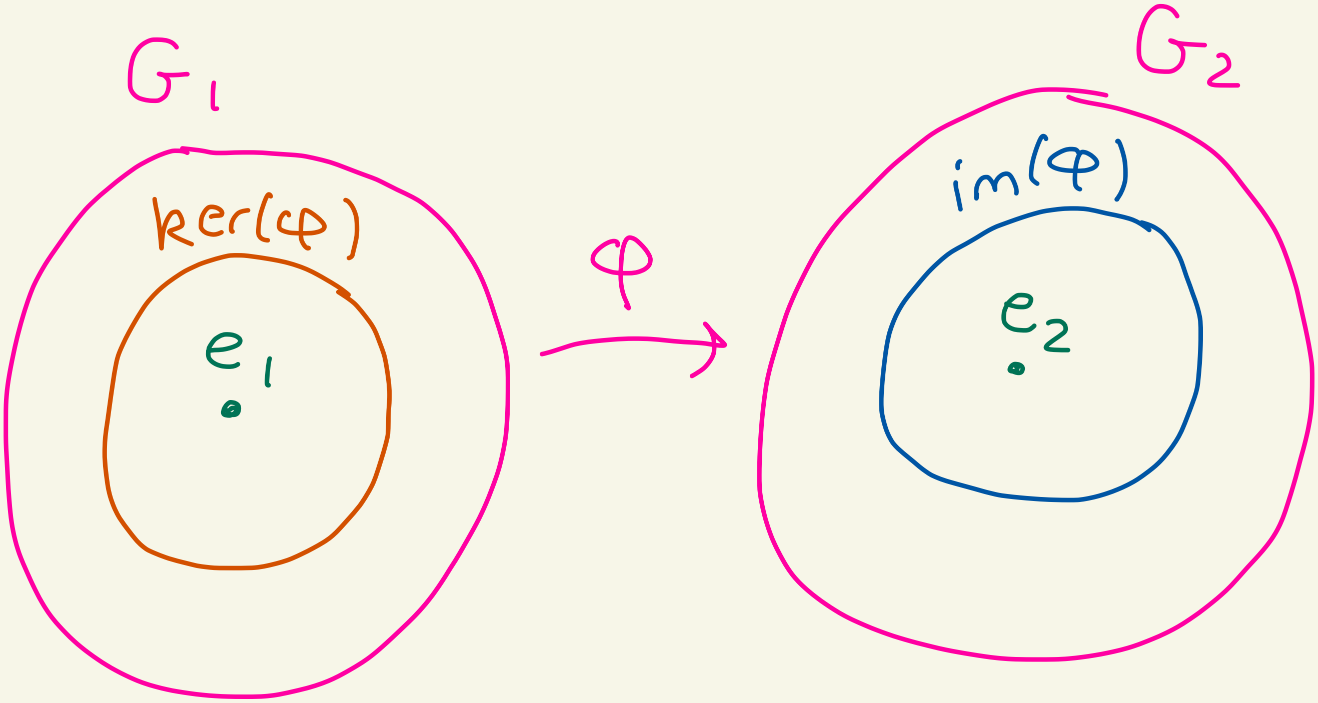
Math 4550

9/29/25

Theorem: Let $\varphi : G_1 \to G_2$ be a homomorphism. Let e_1 and e_2 be the identity elements of G_1 and G_2.



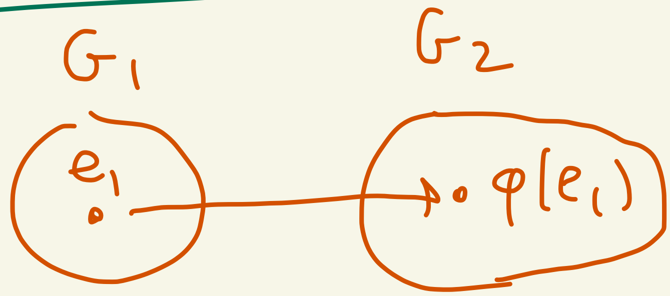
Then:

1. $\varphi(e_1) = e_2$
2. If $x \in G_1$, then $\varphi(x^{-1}) = [\varphi(x)]^{-1}$
3. If $x \in G$ and $k \in \mathbb{Z}$, then $\varphi(x^k) = [\varphi(x)]^k$
4. $\ker(\varphi) \leq G_1$

⑤ $\operatorname{im}(\varphi) \leq G_2$

⑥ φ is one-to-one iff $\ker(\varphi) = \{e_1\}$

⑦ φ is onto iff $\operatorname{im}(\varphi) = G_2$



proof:

① We have

$$\varphi(e_1) \underset{\boxed{e_1 x = x}}{=} \varphi(e_1 e_1) \underset{\boxed{\varphi \text{ homomorphism}}}{=} \varphi(e_1)\varphi(e_1)$$

Then,

$$\underbrace{\varphi(e_1)^{-1}\varphi(e_1)}_{e_2} = \underbrace{\varphi(e_1)^{-1}\varphi(e_1)}_{e_2}\varphi(e_1)$$

So,

$$e_2 = e_2\,\varphi(e_1)$$

Thus, $e_2 = \varphi(e_1)$

e_2 is identity
$e_2 x = x$

(2) We have that

$$\varphi(x^{-1})\varphi(x) = \varphi(x^{-1}x) = \varphi(e_1) \overset{(1)}{=} e_2$$

$\boxed{\varphi \text{ is a homomorphism}}$

So, $\varphi(x^{-1})\varphi(x) = e_2$

Thus, $[\varphi(x)]^{-1} = \varphi(x^{-1})$

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(3) If $k=0$, then

$$\varphi(x^0) = \varphi(e_1) \overset{(1)}{=} e_2 = [\varphi(x)]^0$$

If $k>0$, then

$\boxed{\varphi \text{ is a homomorphism}}$

$$\varphi(x^k) = \varphi(\underbrace{xx\cdots x}_{k\text{ times}}) = \varphi(x)\varphi(x)\cdots\varphi(x) = [\varphi(x)]^k$$

$$\varphi(x^{-k}) = \varphi((x^k)^{-1})$$

$$\overset{②}{=} [\varphi(x^k)]^{-1}$$

φ hom.

$$= [\varphi(x)\varphi(x)\cdots\varphi(x)]^{-1}$$

$$= \varphi(x)^{-1}\varphi(x)^{-1}\cdots\varphi(x)^{-1}$$

$$= [\varphi(x)^{-1}]^k$$

$$= [\varphi(x)]^{-k}$$

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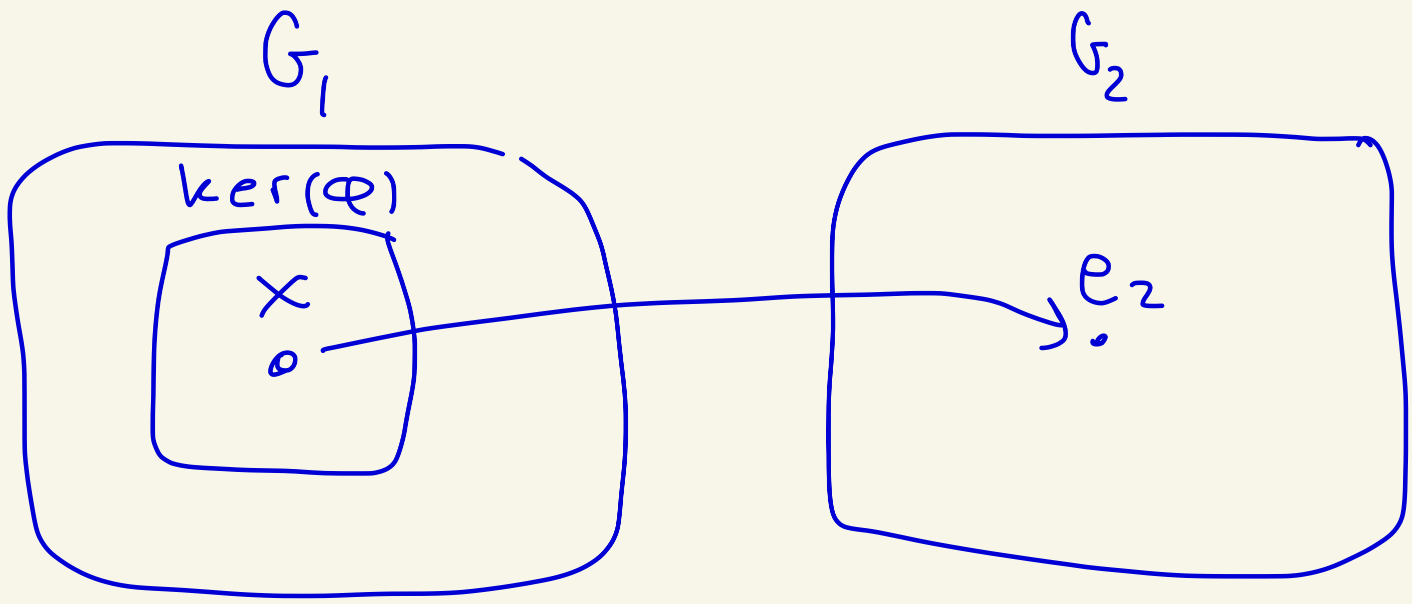
④ Let's show $\ker(\varphi) \leq G_1$.

Recall

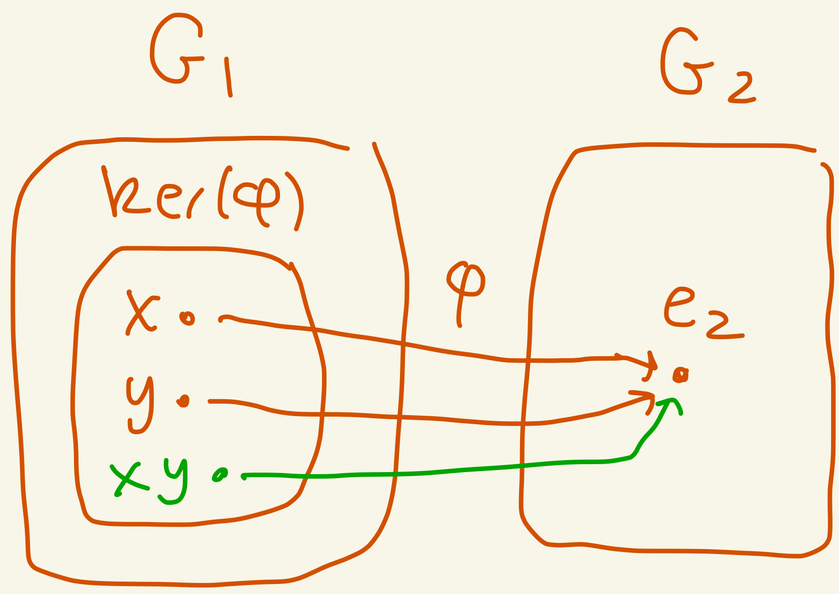
$$\ker(\varphi) = \{x \in G_1 \mid \varphi(x) = e_2\}$$



(i) We know from above that $\varphi(e_1) = e_2$.

So, $e_1 \in \ker(\varphi)$.

(ii) Let $x, y \in \ker(\varphi)$.



Then,

$\varphi(x) = e_2$

and

$\varphi(y) = e_2$

So,

$$\varphi(xy) \underset{\boxed{\varphi \text{ is a hom.}}}{=} \varphi(x)\varphi(y) = e_2 e_2 = e_2$$

Thus, $xy \in \ker(\varphi)$.

(iii) Let $z \in \ker(\varphi)$.

Then, $\varphi(z) \underset{\boxed{\text{def of ker}}}{=} e_2$.

So, $\varphi(z^{-1}) \overset{2}{=} [\varphi(z)]^{-1} = e_2^{-1} \underset{\boxed{e_2 e_2 = e_2}}{=} e_2$

Thus, $z^{-1} \in \ker(\varphi)$.

By (i), (ii), (iii)

$\ker(\varphi) \leq G_1$.

(5) See online notes

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(6) We show φ is 1-1 iff $\ker(\varphi) = \{e_1\}$.

($\Leftarrow$) Suppose $\ker(\varphi) = \{e_1\}$.

Let's show φ is 1-1.

Let $x, y \in G_1$ with $\varphi(x) = \varphi(y)$

We must show that $x = y$.

We have $\varphi(x) = \varphi(y)$.

So, $\varphi(x)\varphi(y)^{-1} = \varphi(y)\varphi(y)^{-1}$

Then $\varphi(x)\varphi(y)^{-1} = e_2$

So, $\varphi(x)\varphi(y^{-1}) = e_2$

(2) φ is

And, $\varphi(xy^{-1}) = e_2$ ⟵ a hom.

So, $xy^{-1} \in \ker(\varphi)$.

Since $\ker(\varphi) = \{e_1\}$ we know $xy^{-1} = e_1$.

Then, $x \underbrace{y^{-1} y}_{e_1} = \underbrace{e_1 y}_{y}$.

So, $x = y$.

Hence, φ is 1-1.

($\Rightarrow$) Suppose φ is 1-1.

We must show $\ker(\varphi) = \{e_1\}$.

From ①, $\boxed{\{e_1\} \subseteq \ker(\varphi)}$.

Now we show $\ker(\varphi) \subseteq \{e_1\}$.

Let $x \in \ker(\varphi)$.

Then, $\varphi(x) = e_2$.

From ① we know $\varphi(e_1) = e_2$.

Thus, $\varphi(x) = \varphi(e_1)$.

Since φ is 1-1 we get that $x = e_1$.

So, $\boxed{\ker(\varphi) \subseteq \{e_1\}.}$

Thus, $\ker(\varphi) = \{e_1\}$.

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⑦ φ is onto iff $\operatorname{im}(\varphi) = G_2$ is the definition of onto.
