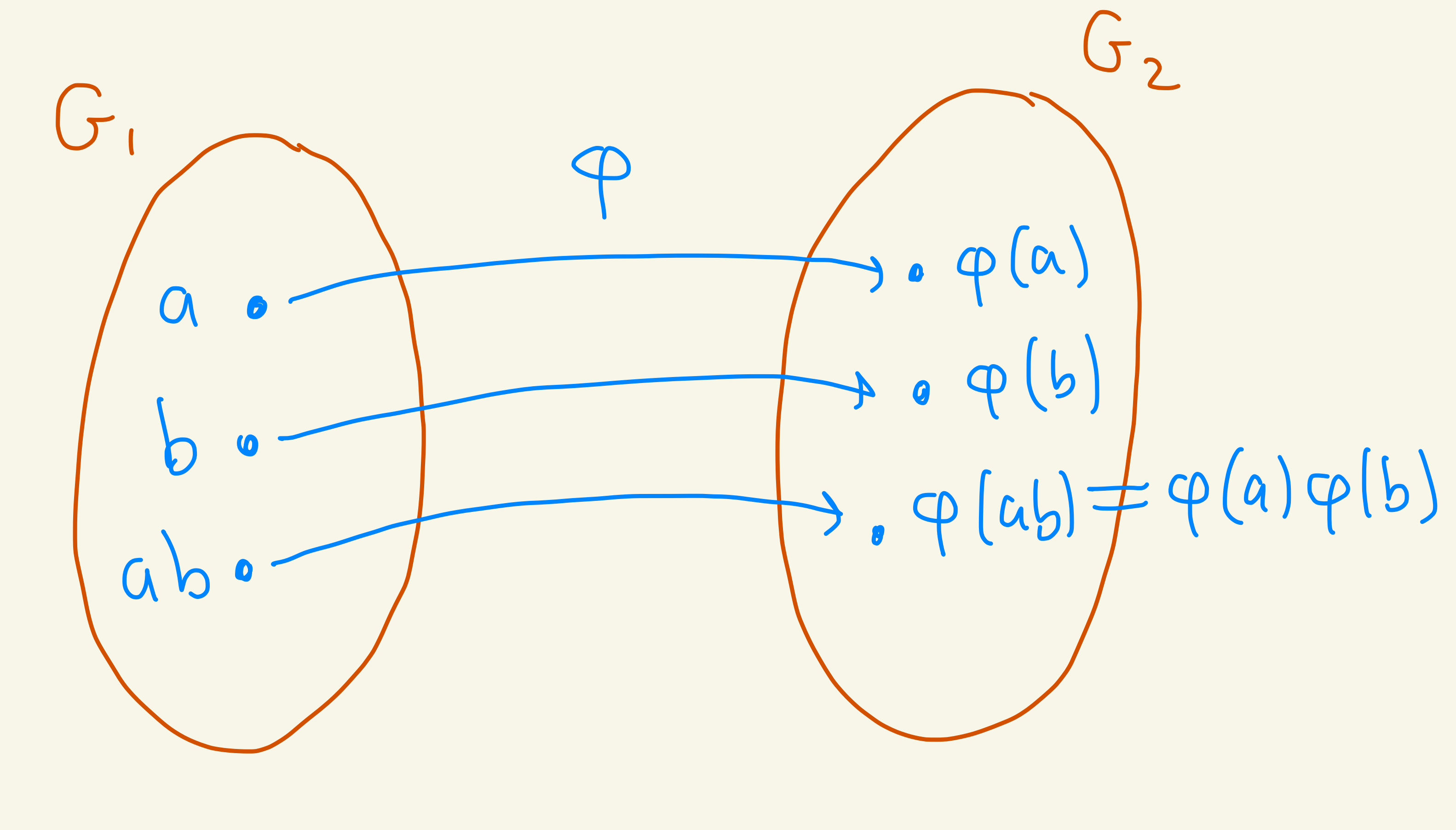
Math 4550
9/24/25

Topic 4 – Homomorphisms

<u>Def:</u> Let G_1 and G_2 be groups. A function $\varphi : G_1 \to G_2$ is called a <u>homomorphism</u> if

$$\varphi(ab) = \varphi(a)\varphi(b)$$

for all $a, b \in G_1$.



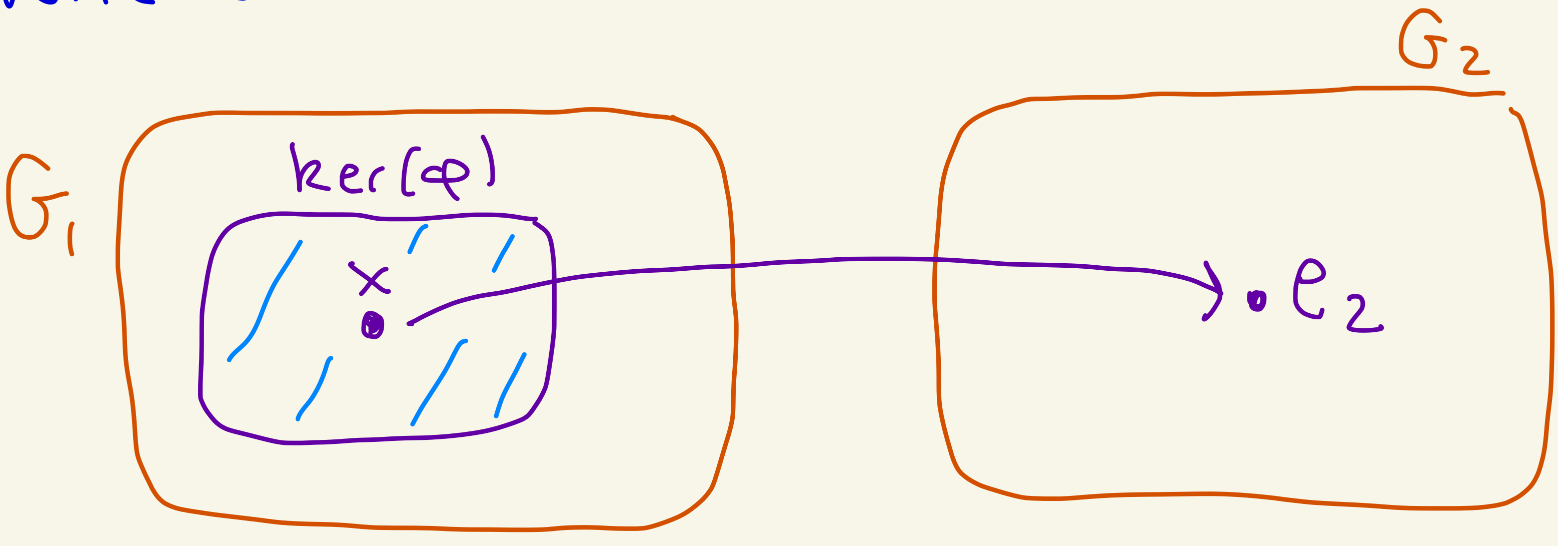
Note: If G_1 has operation $*_1$ and G_2 has operation $*_2$ then the homomorphism condition can be written

$$\varphi(a *_1 b) = \varphi(a) *_2 \varphi(b)$$

The kernel of φ is

$$\ker(\varphi) = \{ x \in G_1 \mid \varphi(x) = e_2 \}$$

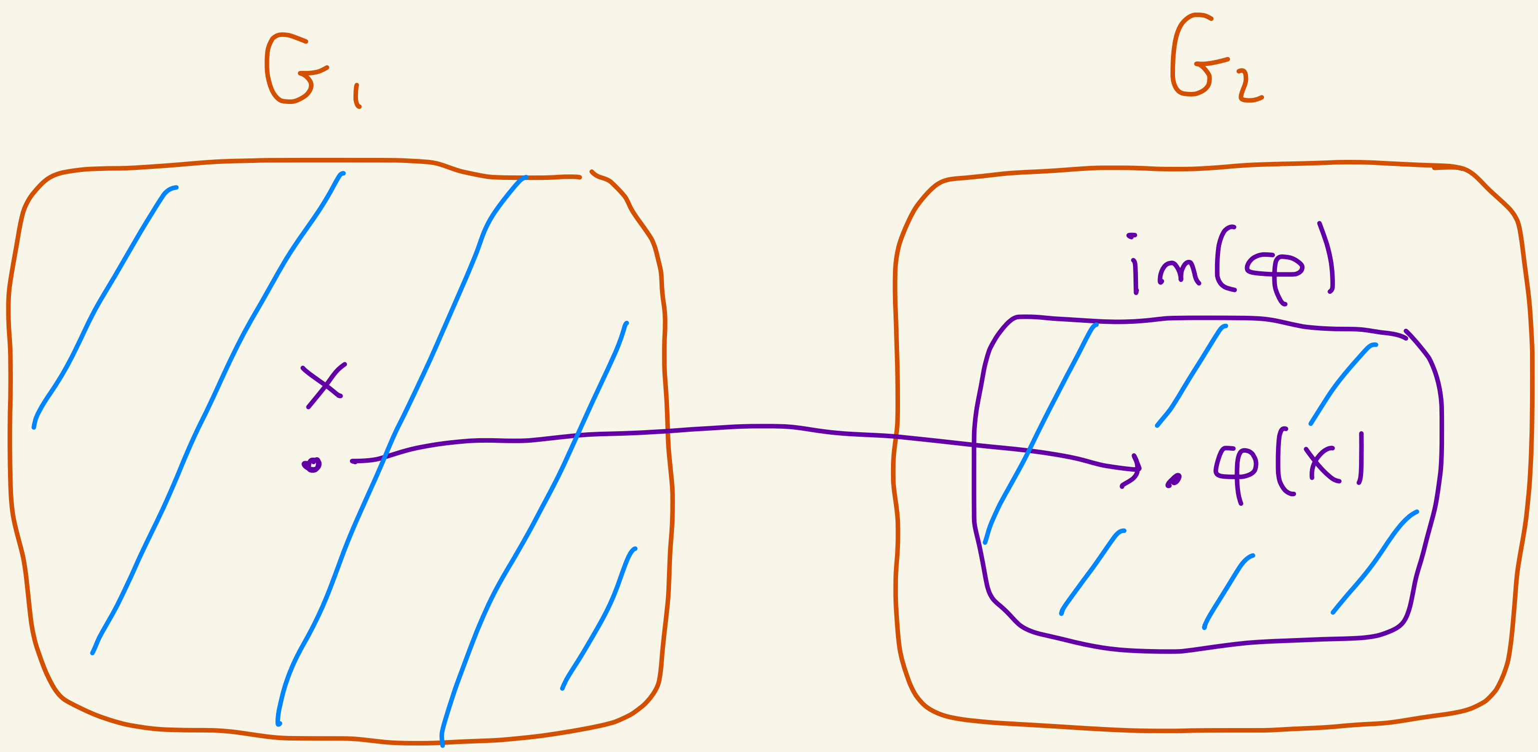
where e_2 is the identity of G_2



The *image* of φ is

$$\mathrm{im}(\varphi) = \{\varphi(x) \mid x \in G_1\}$$

range of φ



If φ is one-to-one and onto, then we call φ an *isomorphism* and we say that G_1 and G_2 are *isomorphic* and write $G_1 \cong G_2$.

Ex: Consider the groups $\mathbb{Z}_3$ and U_3.

addition

multiplication

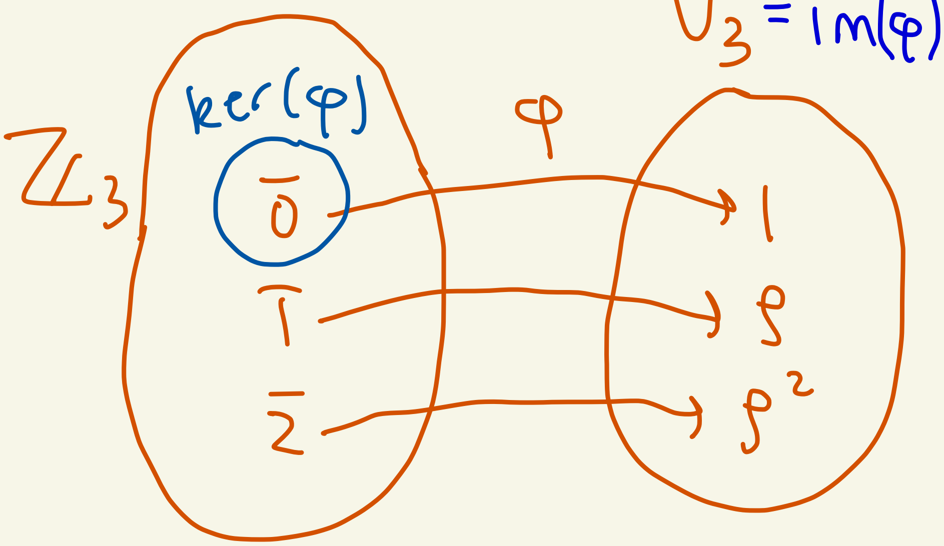
USE: $\boxed{\rho^3 = 1}$

$\mathbb{Z}_3$	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{0}$	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{0}$
$\bar{2}$	$\bar{2}$	$\bar{0}$	$\bar{1}$

U_3	1	ρ	ρ^2
1	1	ρ	ρ^2
ρ	ρ	ρ^2	1
ρ^2	ρ^2	1	ρ

Define $\varphi : \mathbb{Z}_3 \to U_3$ by

$$\varphi(\bar{0}) = 1$$
$$\varphi(\bar{1}) = \rho$$
$$\varphi(\bar{2}) = \rho^2$$



Here φ is 1-1 and onto.

Let's check that φ is a homomorphism.

Show $\varphi(\bar{x}+\bar{y}) = \varphi(\bar{x})\varphi(\bar{y})$ for all $\bar{x}, \bar{y} \in \mathbb{Z}_3$.

We have:

$$\varphi(\bar{0}+\bar{0}) = \varphi(\bar{0}) = 1 = 1 \cdot 1 = \varphi(\bar{0})\varphi(\bar{0})$$
$$\varphi(\bar{0}+\bar{1}) = \varphi(\bar{1}) = \rho = 1 \cdot \rho = \varphi(\bar{0})\varphi(\bar{1})$$
$$\varphi(\bar{0}+\bar{2}) = \varphi(\bar{2}) = \rho^2 = 1 \cdot \rho^2 = \varphi(\bar{0})\varphi(\bar{2})$$
$$\varphi(\bar{1}+\bar{0}) = \varphi(\bar{1}) = \rho = \rho \cdot 1 = \varphi(\bar{1})\varphi(\bar{0})$$
$$\varphi(\bar{1}+\bar{1}) = \varphi(\bar{2}) = \rho^2 = \rho \cdot \rho = \varphi(\bar{1})\varphi(\bar{1})$$
$$\varphi(\bar{1}+\bar{2}) = \varphi(\bar{0}) = 1 = \rho \cdot \rho^2 = \varphi(\bar{1})\varphi(\bar{2})$$
$$\varphi(\bar{2}+\bar{0}) = \varphi(\bar{2}) = \rho^2 = \rho^2 \cdot 1 = \varphi(\bar{2})\varphi(\bar{0})$$
$$\varphi(\bar{2}+\bar{1}) = \varphi(\bar{0}) = 1 = \rho^2 \cdot \rho = \varphi(\bar{2})\varphi(\bar{1})$$
$$\varphi(\bar{2}+\bar{2}) = \varphi(\bar{1}) = \rho = \rho^2 \cdot \rho^2 = \varphi(\bar{2})\varphi(\bar{2})$$

Thus, φ is a homomorphism.

Since φ is 1-1 and onto, φ is an isomorphism.

So, $\mathbb{Z}_3 \cong U_3$. ← $\mathbb{Z}_3$ is isomorphic to U_3

Ex: Let $\varphi : \mathbb{Z} \to \mathbb{Z}$ be defined by $\varphi(x) = 2x$

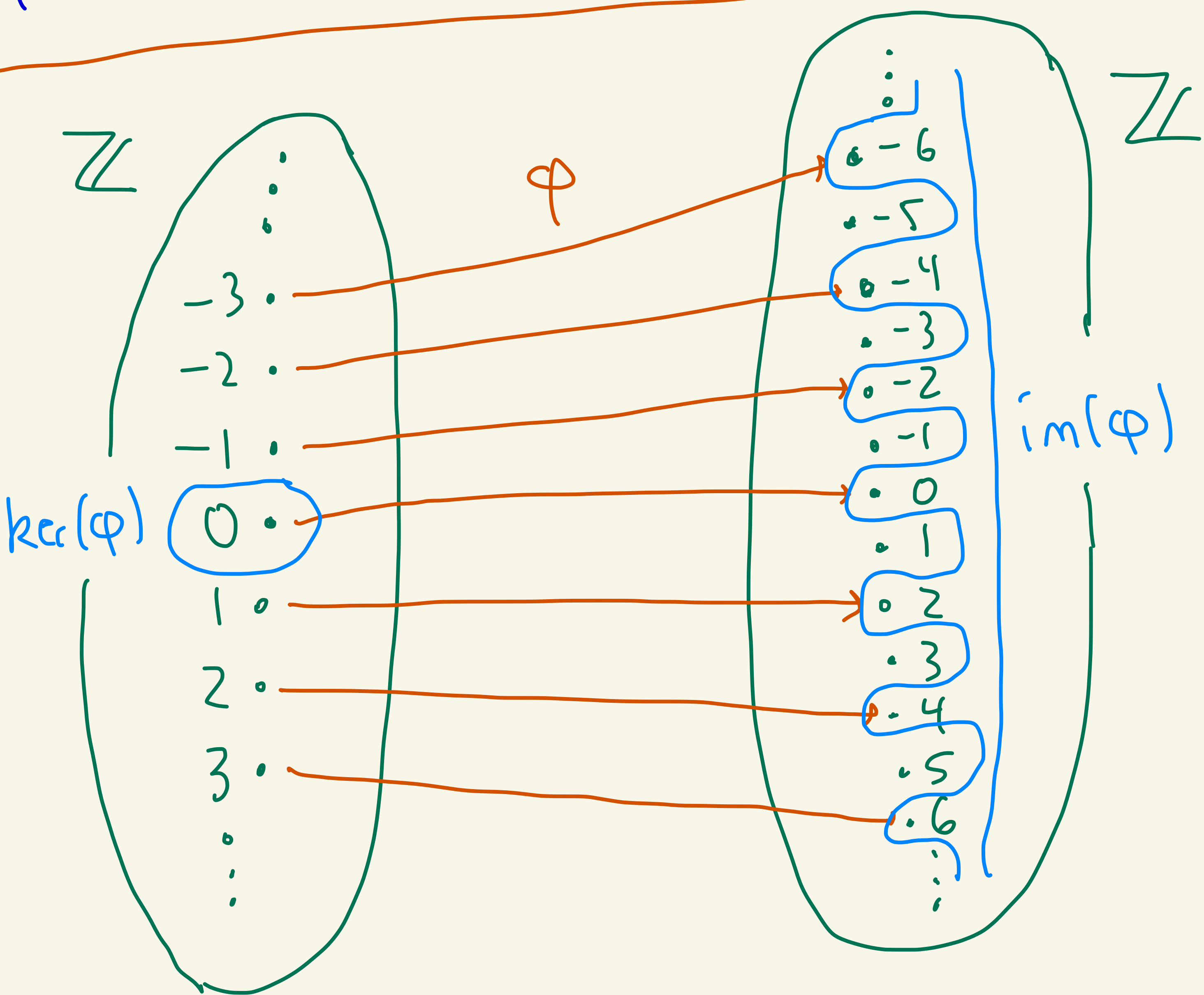
Claim: φ is a homomorphism.

Proof:

Let $x, y \in \mathbb{Z}$.

Then,

$$\varphi(x+y) = 2(x+y) = 2x+2y = \varphi(x)+\varphi(y).$$



$\ker(\varphi) = \{0\}$

$\operatorname{im}(\varphi) = \underbrace{\langle 2 \rangle}_{\text{also called } 2\mathbb{Z}} = \{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}$

φ is 1-1:

Suppose $\varphi(x) = \varphi(y)$.

Then, $2x = 2y$.

So, $x = y$.

f is 1-1 if $f(x) = f(y)$ implies $x = y$

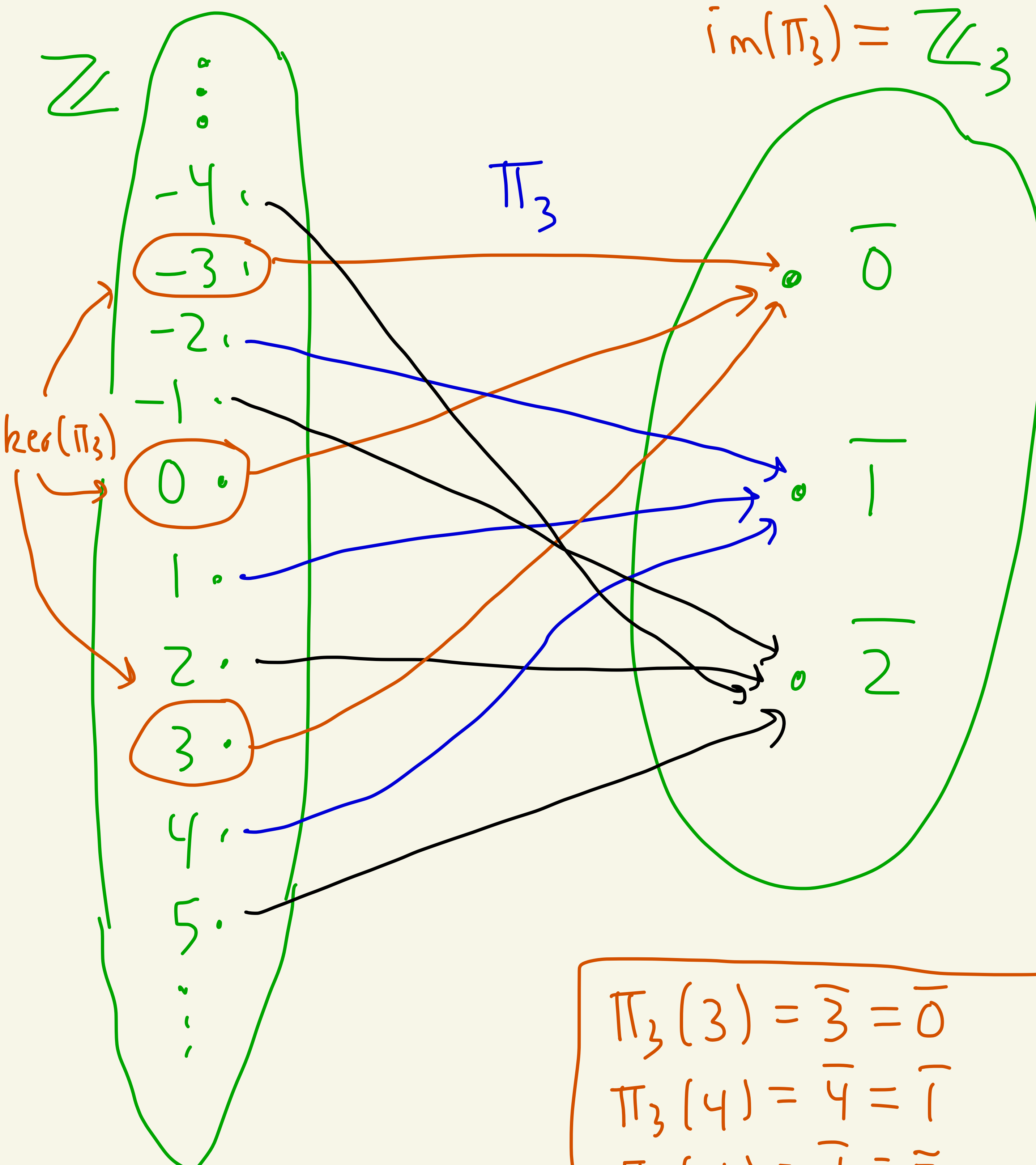
φ is not onto since $\operatorname{im}(\varphi) \neq \mathbb{Z}$

$f : A \to B$

f is onto if $\operatorname{im}(f) = B$

φ is a homomorphism but not an isomorphism

Ex: Define $\pi_3 : \mathbb{Z} \to \mathbb{Z}_3$ by $\pi_3(x) = \overline{x}$.



$\pi_3(3) = \overline{3} = \overline{0}$

$\pi_3(4) = \overline{4} = \overline{1}$

$\pi_3(-1) = \overline{-1} = \overline{2}$

π_3 is not 1-1.

π_3 is onto.

$$\boxed{\begin{array}{l}\underline{\pi_3 \text{ is a homomorphism}}: \\ \pi_3(x+y) = \overline{x+y} = \bar{x} + \bar{y} = \pi_3(x) + \pi_3(y)\end{array}}$$

$$\ker(\pi_3) = \langle 3 \rangle = \{\ldots, -6, -3, 0, 3, 6, \ldots\}$$
$$= 3\mathbb{Z}$$

$$\operatorname{im}(\pi_3) = \mathbb{Z}_3$$
