Math 4550
9/17/25

TOPIC 3 – DIRECT PRODUCTS

Def: Let G_1, G_2 be groups. The *Cartesian product* is

$$G_1 \times G_2 = \{(g_1, g_2) \mid g_1 \in G_1, g_2 \in G_2\}$$

Ex: $U_3 = \{1, \rho, \rho^2\}$

$\mathbb{Z}_2 = \{\overline{0}, \overline{1}\}$

$$U_3 \times \mathbb{Z}_2 = \{(1,\overline{0}), (1,\overline{1}), (\rho,\overline{0}), (\rho,\overline{1}), (\rho^2,\overline{0}), (\rho^2,\overline{1})\}$$

We can make this into a group.

Operation:

$$(a,b)(c,d) = (ac,\, b+d)$$

$\mathbb{Z}_2$

add

U_3

multiply

in U_3

in $\mathbb{Z}_2$

For example:

$$(1,\bar{1})(\rho^2,\bar{1}) = (1\cdot\rho^2,\, \bar{1}+\bar{1})$$

$$= (\rho^2,\bar{2})$$

$$= (\rho^2,\bar{0})$$

identity: $(1,\bar{0})$

identity from U_3

identity from $\mathbb{Z}_2$

For example,

$$\underbrace{(1,\bar{0})}_{\text{identity}}(\rho,\bar{1}) = (1\cdot\rho, \bar{0}+\bar{1})$$
$$= (\rho, \bar{1})$$

<u>Theorem:</u> Let G_1, G_2 be groups with identity elements e_1, e_2 respectively. Then $G_1 \times G_2$ is a group using the operation

$$(a,b)(c,d) = (\underbrace{ac}_{\boxed{\text{operation in } G_1}}, \underbrace{bd}_{\boxed{\text{operation in } G_2}})$$

The identity element is (e_1, e_2)

The inverse of (a,b) is (a^{-1}, b^{-1})

Proof:

① (closure)

Let $(a,b), (c,d) \in G_1 \times G_2$.

Then, $a, c \in G_1$ and $b, d \in G_2$.

Since G_1 is a group,

we know $ac \in G_1$.

Since G_2 is a group,

we know $bd \in G_2$.

Then,

$(a,b)(c,d) = (ac, bd) \in G_1 \times G_2$.

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② (associativity)

Let $(a,b), (c,d), (e,f) \in G_1 \times G_2$.

Then,

$$(a,b)\big[(c,d)(e,f)\big]$$

$$= (a,b)(ce,df)$$

$$= (a(ce),\ b(df))$$

$$= ((ac)e,\ (bd)f)$$

$$= (ac,bd)(e,f)$$

$$= \big[(a,b)(c,d)\big](e,f)$$

G_1 is associative
G_2 is associative

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③ (identity)

Let's show that (e_1, e_2) is the identity of $G_1 \times G_2$

where e_1 is the identity of G_1 and e_2 is the identity of G_2.

Let $(a, b) \in G_1 \times G_2$.

$a \in G_1$ $b \in G_2$

Then,

$(a,b)(e_1, e_2) = (ae_1, be_2) = (a, b)$

$(e_1, e_2)(a, b) = (e_1 a, e_2 b) = (a, b)$.

So, (e_1, e_2) is the identity of $G_1 \times G_2$

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(4) (inverses)

Let $(a, b) \in G_1 \times G_2$.

Then $a \in G_1$ and $b \in G_2$

Since G_1 is a group, we know a^{-1} exists in G_1 and $aa^{-1} = a^{-1}a = e_1$

Since G_2 is a group, we know b^{-1} exists in G_2 and $bb^{-1} = b^{-1}b = e_2$.

Then, $(a^{-1}, b^{-1}) \in G_1 \times G_2$.

And,

$$(a,b)(a^{-1},b^{-1}) = (aa^{-1}, bb^{-1}) = (e_1, e_2)$$

$$(a^{-1},b^{-1})(a,b) = (a^{-1}a, b^{-1}b) = (e_1, e_2)$$

Thus, (a^{-1}, b^{-1}) is the inverse of (a,b).

By ①, ②, ③, ④ $G_1 \times G_2$ is a group $\blacksquare$