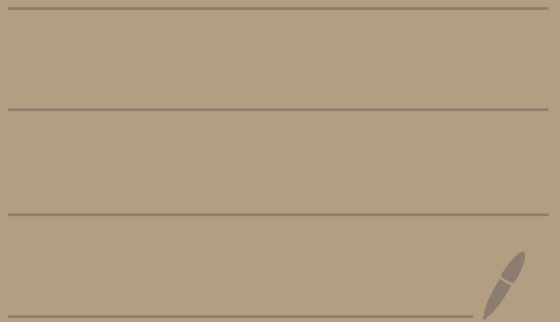


Math 4550

9/17/25



TOPIC 3 - DIRECT PRODUCTS

Def: Let G_1, G_2 be groups.

The Cartesian product is

$$G_1 \times G_2 = \{ (g_1, g_2) \mid g_1 \in G_1, g_2 \in G_2 \}$$

Ex: $U_3 = \{1, \rho, \rho^2\}$

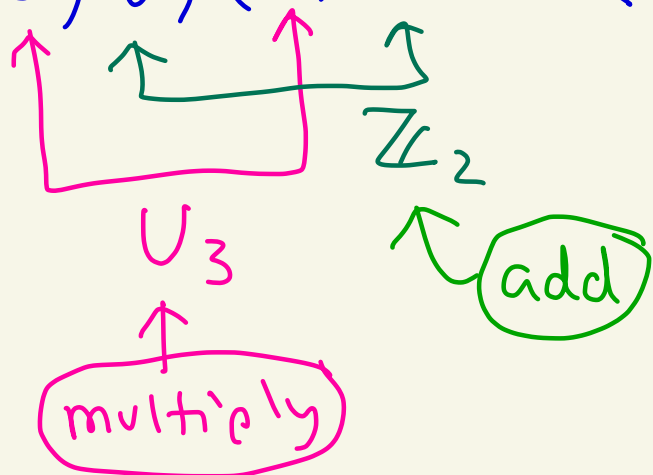
$$\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$$

$$U_3 \times \mathbb{Z}_2 = \{ (1, \bar{0}), (1, \bar{1}), (\rho, \bar{0}), (\rho, \bar{1}), (\rho^2, \bar{0}), (\rho^2, \bar{1}) \}$$

We can make this into a group.

Operation:

$$(a, b)(c, d) = (ac, b + d)$$



For example:

$$\begin{aligned}(1, \overline{1})(\overline{9}^2, \overline{1}) &= (\overline{1 \cdot 9}^2, \overline{1} + \overline{1}) \\ &= (\overline{9}^2, \overline{2}) \\ &= (\overline{9}^2, \overline{0})\end{aligned}$$

identity: $(1, \overline{0})$

identity
from
 U_3

identity from \mathbb{Z}_2

For example,

$$\underbrace{(1, \bar{0})}_{\text{identity}} (\bar{s}, \bar{T}) = (1 \cdot \bar{s}, \bar{0} + \bar{T}) \\ = (\bar{s}, \bar{T})$$

Theorem: Let G_1, G_2 be groups with identity elements e_1, e_2 respectively. Then $G_1 \times G_2$ is a group using the operation

$$(a, b)(c, d) = (ac, bd)$$

operation
in G_1

operation
in G_2

The identity element is (e_1, e_2)

The inverse of (a, b) is (a^{-1}, b^{-1})

Proof:

① (closure)

Let $(a, b), (c, d) \in G_1 \times G_2$.

Then, $a, c \in G_1$ and $b, d \in G_2$.

Since G_1 is a group,

we know $ac \in G_1$.

Since G_2 is a group,

we know $bd \in G_2$.

Then,

$$(a, b)(c, d) = (ac, bd) \in G_1 \times G_2.$$

② (associativity)

Let $(a, b), (c, d), (e, f) \in G_1 \times G_2$.

Then,

$$(a,b)[(c,d)(e,f)]$$

$$= (a,b)(ce,df)$$

$$= (a(ce), b(df))$$

$$= ((ac)e, (bd)f)$$

$$= (ac, bd)(e, f)$$

$$= [(a,b)(c,d)](e, f)$$

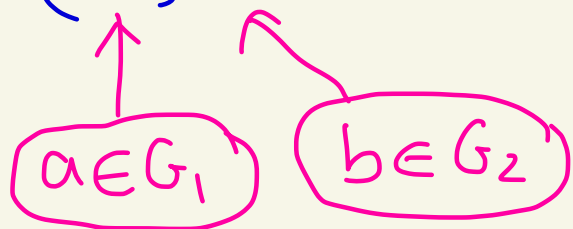
G_1 is associative
 G_2 is associative

③ (identity)

Let's show that (e_1, e_2)
 is the identity of $G_1 \times G_2$

where e_1 is the identity of G_1
and e_2 is the identity of G_2 .

Let $(a, b) \in G_1 \times G_2$.



Then,

$$(a, b)(e_1, e_2) = (ae_1, be_2) = (a, b)$$

$$(e_1, e_2)(a, b) = (e_1a, e_2b) = (a, b).$$

So, (e_1, e_2) is the identity of $G_1 \times G_2$

④ (inverses)

Let $(a, b) \in G_1 \times G_2$.

Then $a \in G_1$ and $b \in G_2$

Since G_1 is a group, we
know a^{-1} exists in G_1
and $aa^{-1} = a^{-1}a = e_1$

Since G_2 is a group, we
know b^{-1} exists in G_2
and $bb^{-1} = b^{-1}b = e_2$.

Then, $(a^{-1}, b^{-1}) \in G_1 \times G_2$.

And,

$$(a, b)(a^{-1}, b^{-1}) = (aa^{-1}, bb^{-1}) = (e_1, e_2)$$

$$(a^{-1}, b^{-1})(a, b) = (a^{-1}a, b^{-1}b) = (e_1, e_2)$$

Thus, (a^{-1}, b^{-1}) is the
inverse of (a, b) .

By ①, ②, ③, ④ $G_1 \times G_2$ is a group 