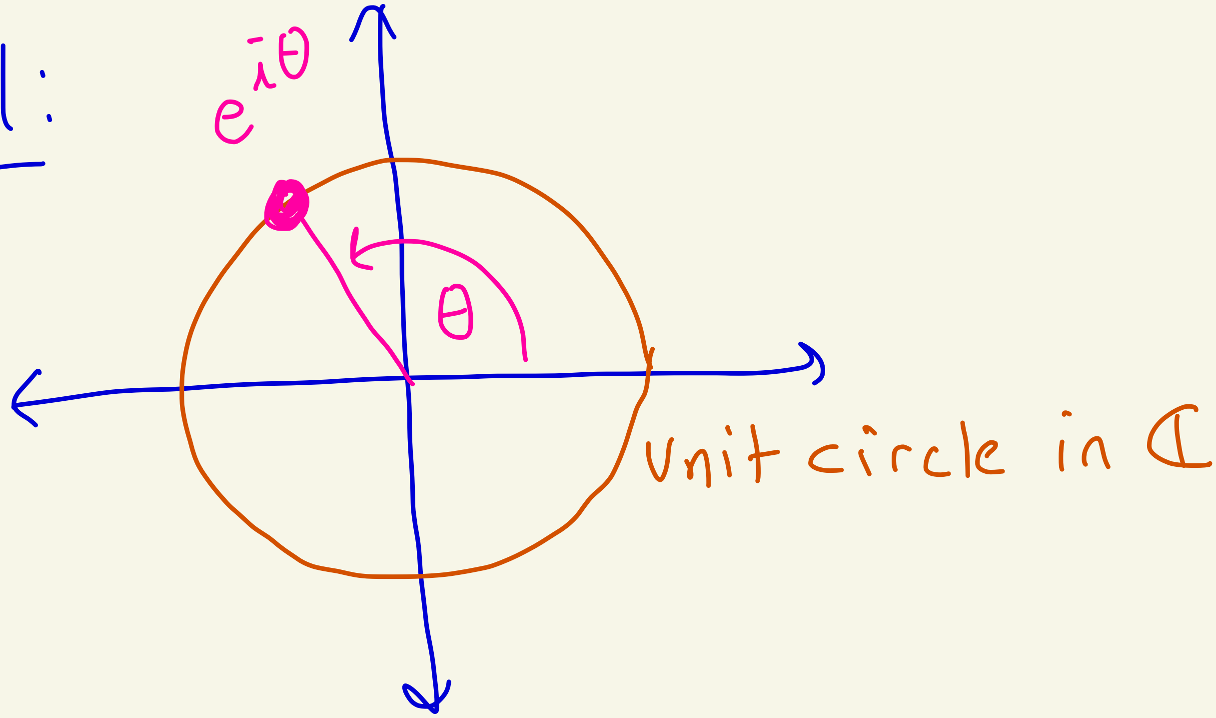
Math 4550

8/27/25

Recall:



$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

Theorem: Let $\theta_1, \theta_2 \in \mathbb{R}$.

Then: $e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1 + \theta_2)}$

proof:

$$e^{i\theta_1} e^{i\theta_2} = [\cos(\theta_1) + i\sin(\theta_1)][\cos(\theta_2) + i\sin(\theta_2)]$$

$$= \cos(\theta_1)\cos(\theta_2) + i\cos(\theta_1)\sin(\theta_2) + i\sin(\theta_1)\cos(\theta_2) + \underbrace{i^2}_{i^2=-1}\sin(\theta_1)\sin(\theta_2)$$

$$= \overbrace{\left[\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)\right]}^{\cos(\theta_1+\theta_2)}$$

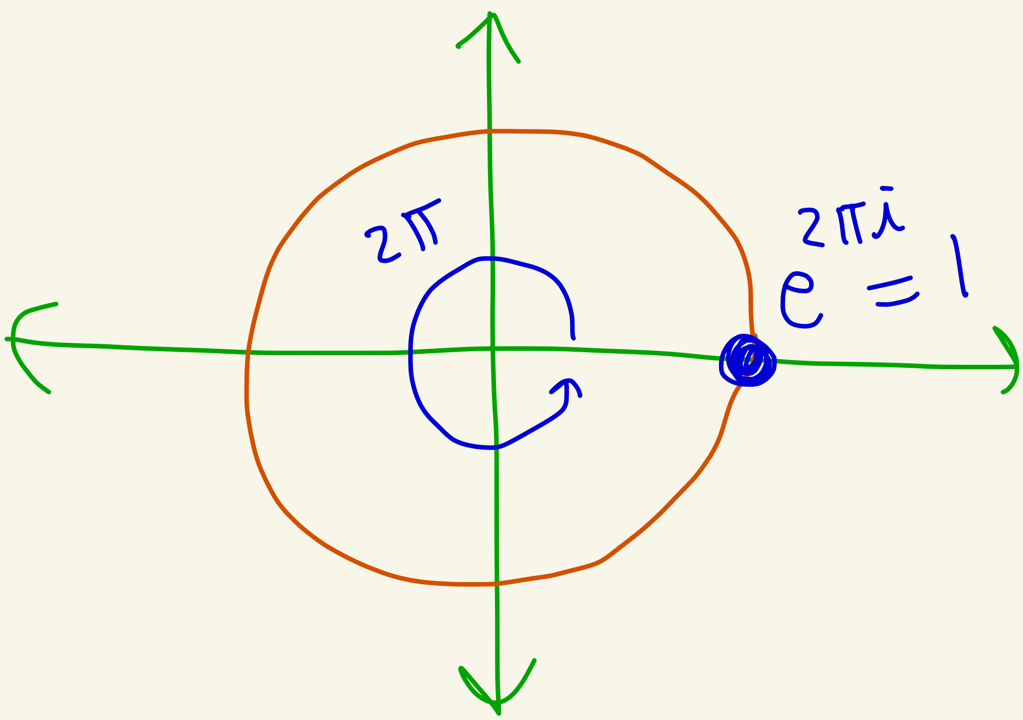
$$+ i\underbrace{\left[\cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2)\right]}_{\sin(\theta_1+\theta_2)}$$

$$= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)$$

$$= e^{i(\theta_1+\theta_2)} \qquad \blacksquare$$

<u>Ex:</u> $e^{\frac{\pi}{2}i} e^{\frac{3\pi}{2}i} = e^{2\pi i} = \cos(2\pi) + i\sin(2\pi)$

$= 1 + 0i = 1$

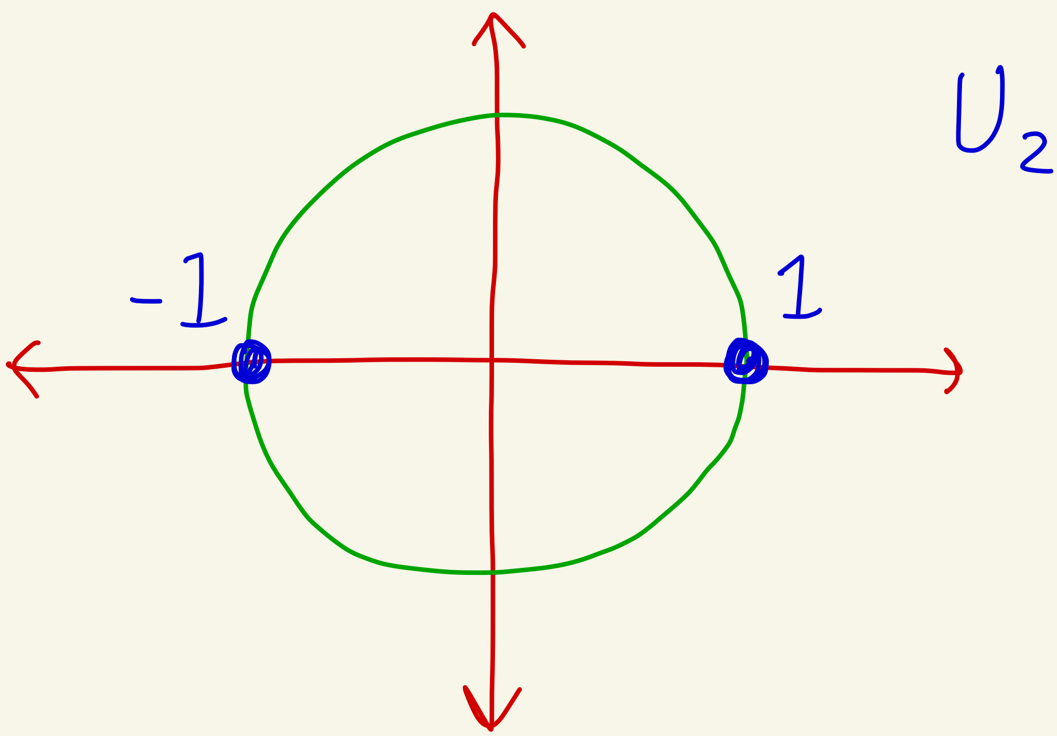


Def: The n-th roots of unity are

$$U_n = \{ z \mid z \in \mathbb{C} \text{ and } z^n = 1 \}$$

Ex: $U_2 = \{ z \mid z \in \mathbb{C} \text{ and } z^2 = 1 \}$

$$= \{ 1, -1 \}$$



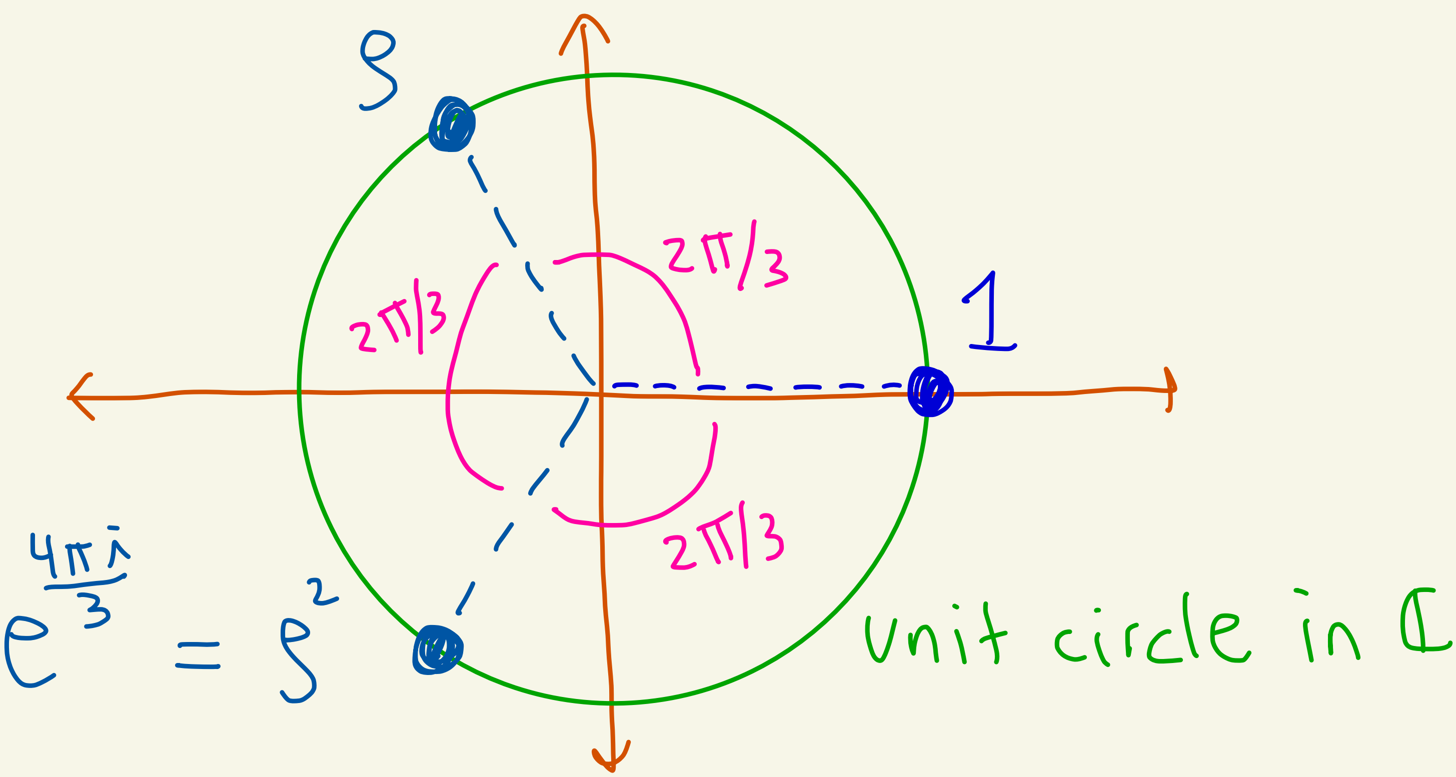
In Math 4680 / Pre-calc you show that

$$U_n = \left\{ \left(e^{\frac{2\pi i}{n}}\right)^k \;\middle|\; k = 0, 1, 2, \ldots, n-1 \right\}$$

$$= \{1, \rho, \rho^2, \ldots, \rho^{n-1}\}$$

where $\rho = e^{2\pi i/n}$

Ex: $U_3 = \{1, \rho, \rho^2\}$ where $\rho = e^{\frac{2\pi i}{3}}$



Theorem: U_n is an abelian group under multiplication. The identity is 1.

ab=ba

proof: See online notes $\blacksquare$

Main computational fact:

In U_n, if $\rho = e^{2\pi i/n}$ then $\rho^n = 1$

pf: $\rho^n = \left(e^{2\pi i/n}\right)^n = e^{2\pi i} = 1$ $\blacksquare$

Group table for U_3: (operation = mult.)

$\boxed{\text{Key fact: } \rho^3 = 1}$

U_3	1	ρ	ρ^2
1	1	ρ	ρ^2
ρ	ρ	ρ^2	1
ρ^2	ρ^2	1	ρ

$\rho \cdot \rho^2 = \rho^3 = 1$

$$\begin{aligned} \rho^2 \rho^2 = \rho^4 &= \rho^3 \rho \\ &= 1 \cdot \rho \\ &= \rho \end{aligned}$$

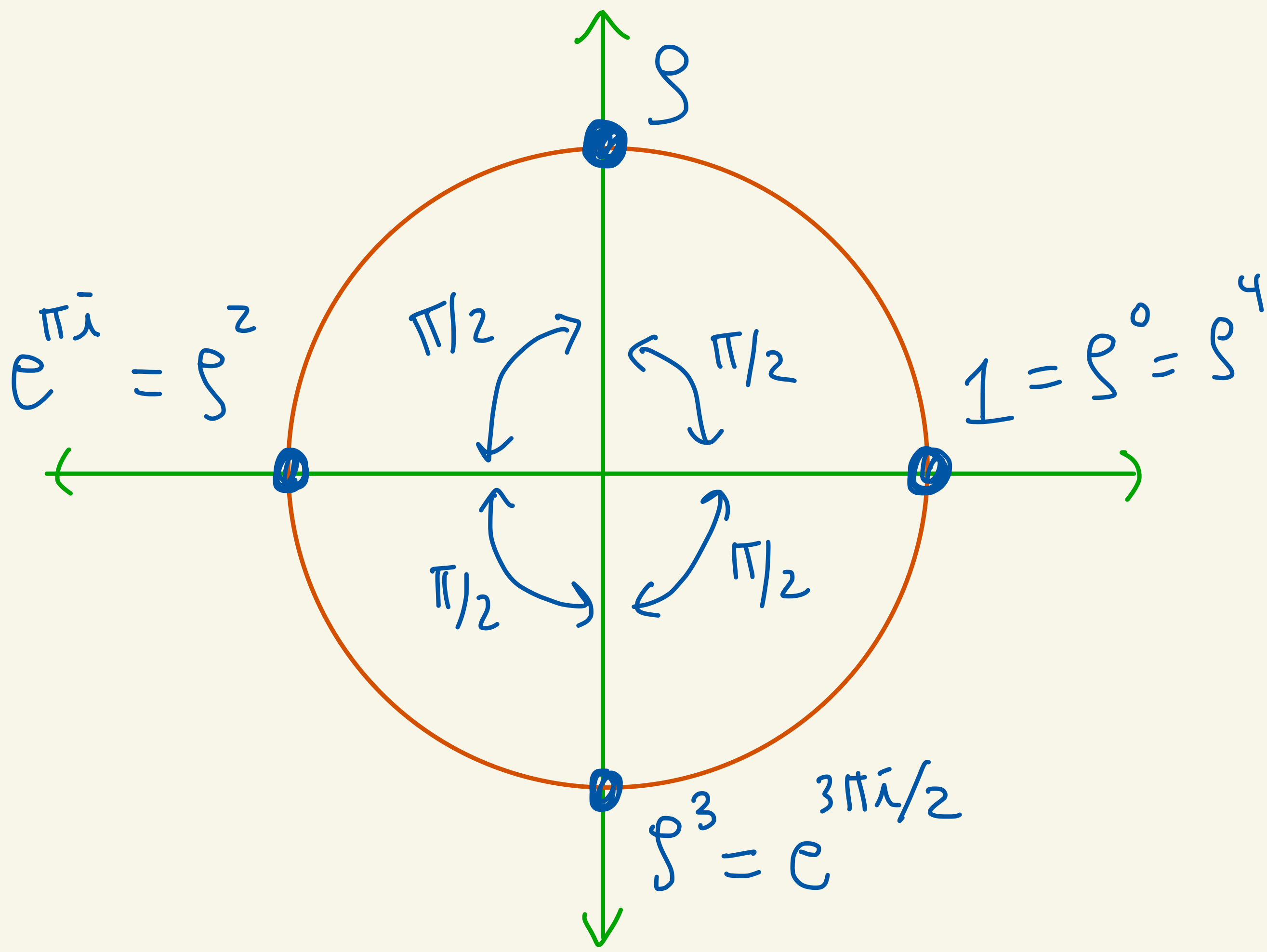
Note: $1^{-1} = 1$

$\rho^{-1} = \rho^2$

$(\rho^2)^{-1} = \rho$

Ex: $U_4 = \{1, \rho, \rho^2, \rho^3\}$

where $\rho = e^{2\pi i/4} = e^{\pi i/2}$



Here: $\rho^4 = 1$

Ex: $\rho^2 \cdot \rho^3 = \rho^5 = \rho^4 \rho = 1 \cdot \rho = \rho$

$\rho^2 \cdot \rho^2 = \rho^4 = 1 \leftarrow (\rho^2)^{-1} = \rho^2$

$\rho \cdot \rho^3 = \rho^4 = 1 \leftarrow \rho^{-1} = \rho^3$

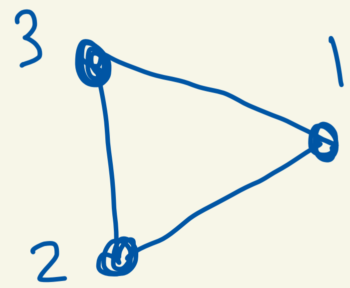
Ex: Dihedral groups

Let $n \geq 3$. The dihedral group D_{2n} is the set of symmetries of the regular n-gon.

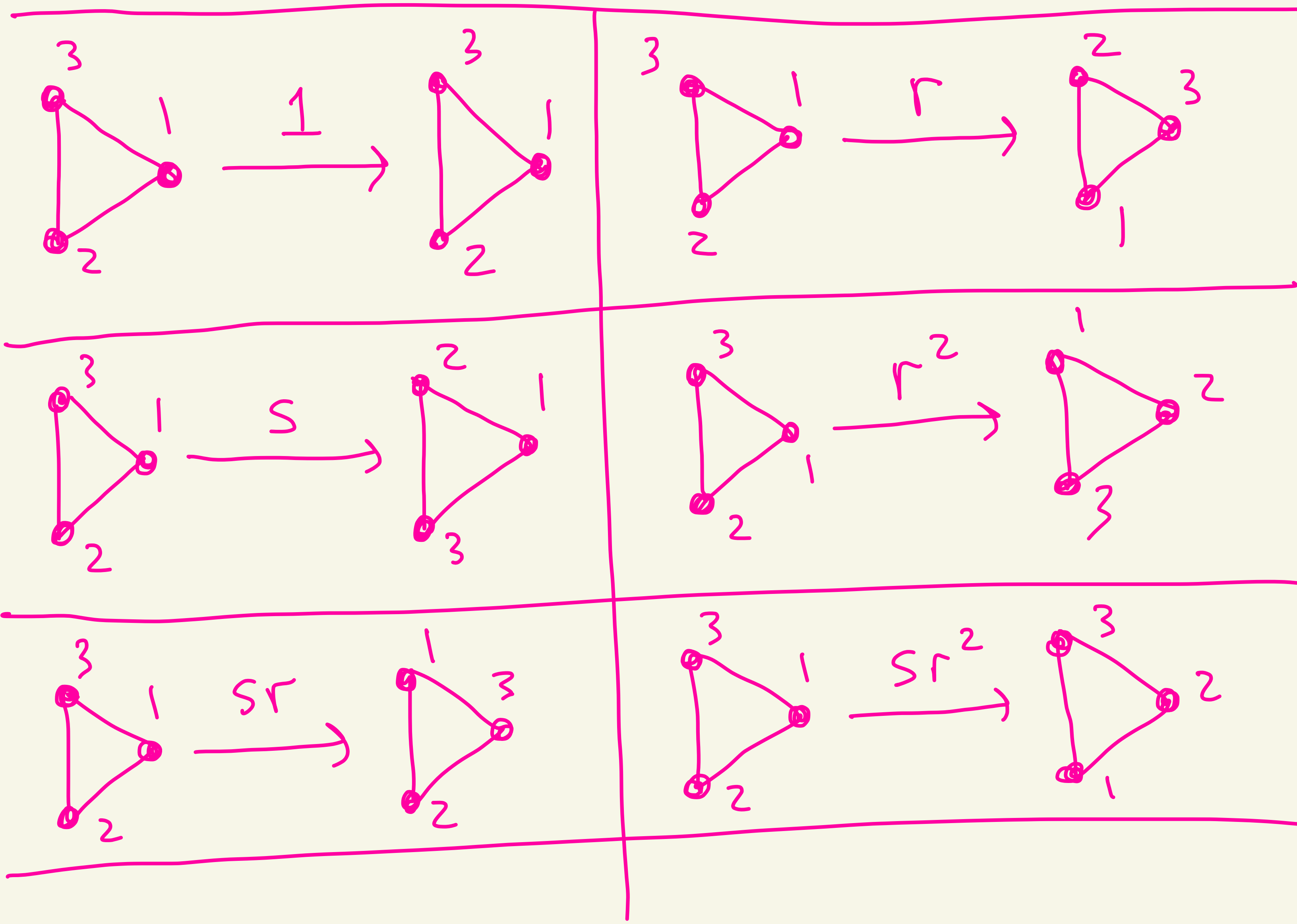
[regular n-gon is an n-sided polygon where each side is the same length]

The group operation is function composition

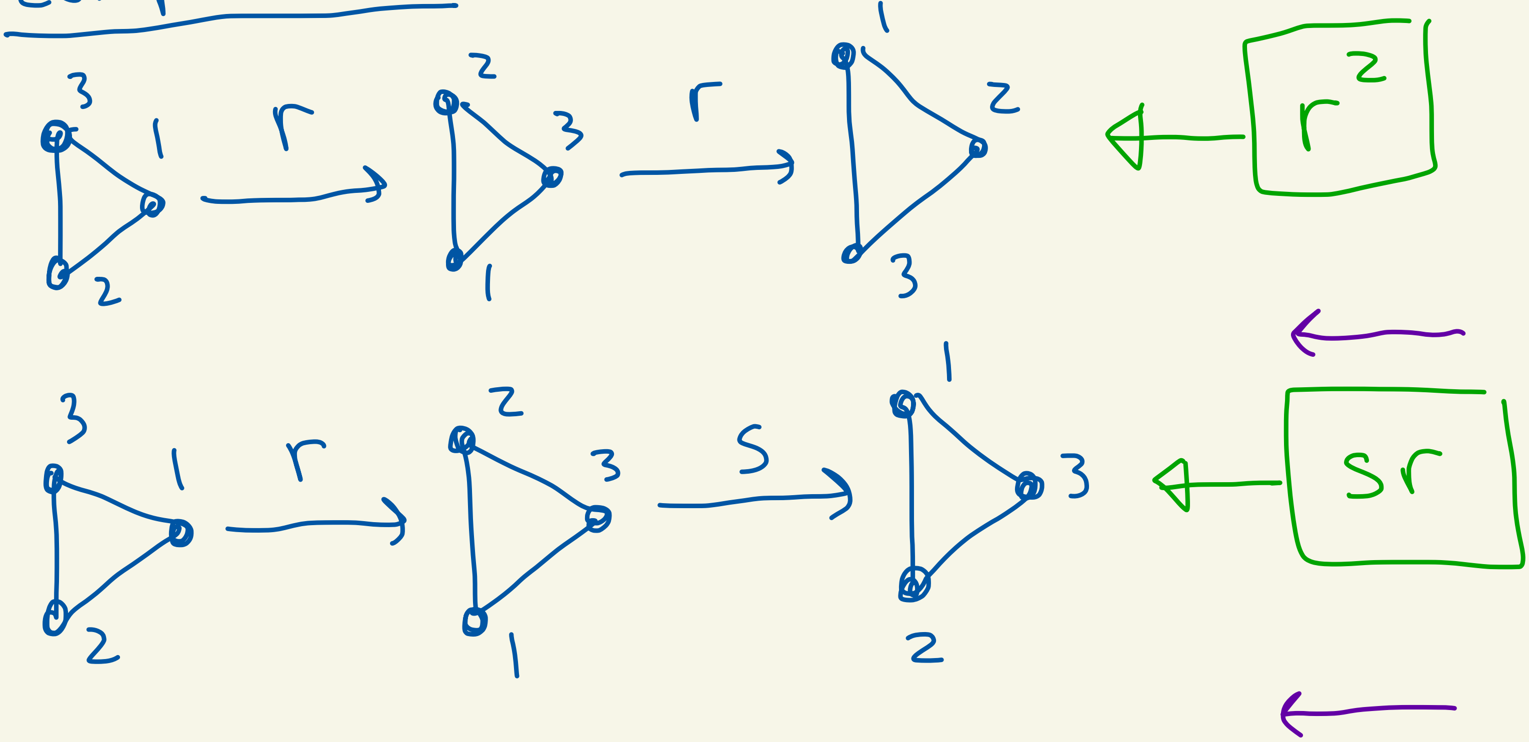
Let's look at $n=3$.
Let's find the elements of $D_6 = D_{2\cdot 3}$, the symmetries of the 3-gon

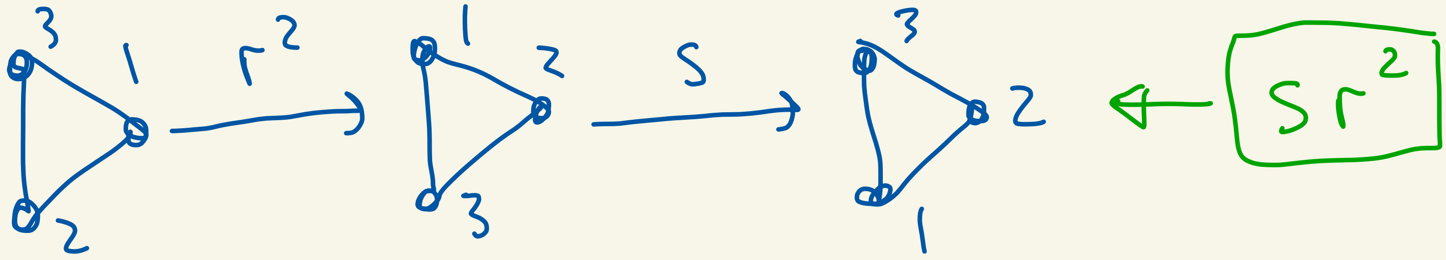


Here are the 6 elements of D_6:



Computations:





So,

$D_6 = \{1, r, r^2, s, sr, sr^2\}$

1 is the identity

$r^3 = 1, \quad s^2 = 1$

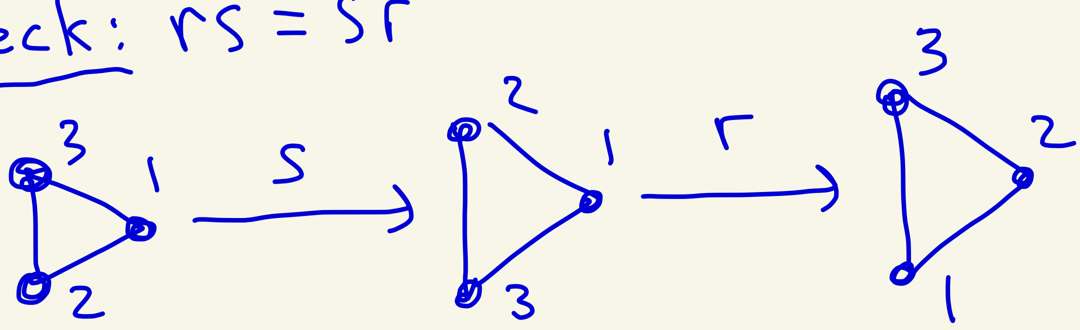
$r^{-1} = r^2$ since $r \cdot r^2 = r^3 = 1$

$s^{-1} = s$ since $s \cdot s = s^2 = 1$

Also,

$rs = sr^{-1} = sr^2$ and $r^2 s = sr^{-2} = sr$

Check: $rs = sr^2$



Some calculations: $D_6 = \{1, r, r^2, s, sr, sr^2\}$

Key facts in D_6 are:

$r^3 = 1$, $s^2 = 1$, $rs = sr^2 = sr^{-1}$

$r^2 s = sr^{-2} = sr$

$r^2 s r^3 = s r^{-2} r^3 = sr$

$(sr)(r^2) = sr^3 = s \cdot 1 = s$

$(r^2)(sr) = sr^{-2} r = sr^{-1} = sr^2$

not equal

D_6 is not abelian since $(sr)(r^2) \neq (r^2)(sr)$