Math 4550

8/25/25

Def: Let $n \in \mathbb{Z}$, $n \geq 2$.
Given $a, b \in \mathbb{Z}$ we write $a \equiv b \pmod{n}$ if $n \mid (a-b)$.

Ex: $n = 4$

$10 \equiv 2 \pmod{4}$

because $10 - 2 = 8 = 4 \cdot 2$.

That is, $4 \mid (10-2)$.

8 apart

8 is a multiple of 4

In Math 3450, you show that mod n is an equivalence relation on $\mathbb{Z}$ and hence the equivalence classes

$$\overline{x} = \{ y \mid y \in \mathbb{Z} \text{ and } y \equiv x \pmod{n} \}$$

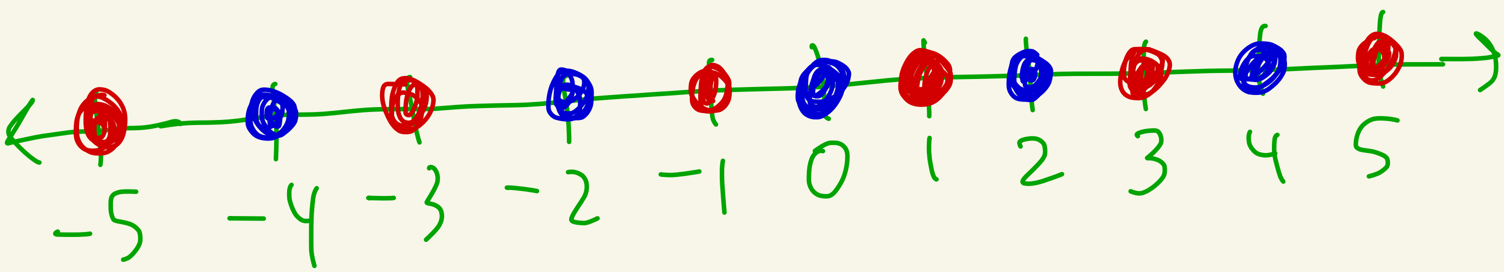
partition $\mathbb{Z}$

Ex: Let $n = 2$

Then:

$$\overline{0} = \{y \mid y \in \mathbb{Z} \text{ and } y \equiv 0 \pmod{2}\}$$
$$= \{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}$$

$$\overline{1} = \{y \mid y \in \mathbb{Z} \text{ and } y \equiv 1 \pmod{2}\}$$
$$= \{\ldots, -5, -3, -1, 1, 3, 5, \ldots\}$$



Note:
$$\overline{3} = \{y \mid y \in \mathbb{Z} \text{ and } y \equiv 3 \pmod{2}\}$$
$$= \{\ldots, -3, -1, 1, 3, 5, 7, \ldots\}$$
$$= \overline{1}$$

and $\overline{-2} = \{y \mid y \in \mathbb{Z} \text{ and } y \equiv -2 \pmod{2}\}$

$= \{\ldots, -6, -4, -2, 0, 2, 4, \ldots\}$

$= \overline{0}$

There are only two unique equivalence classes: $\overline{0}$ and $\overline{1}$

For general $n \geq 2$, the unique equivalence classes are

$\overline{0}, \overline{1}, \overline{2}, \ldots, \overline{n-1}$

We define the set of <u>integers modulo n</u> to be

$$\mathbb{Z}_n = \{\overline{0}, \overline{1}, \overline{2}, \ldots, \overline{n-1}\}$$

One has these facts:

- $\overline{a} = \overline{b}$ iff $a \equiv b \pmod{n}$
- If you divide n into x and get remainder r, then $\overline{x} = \overline{r}$.
- Define
  $\overline{a} + \overline{b} = \overline{a+b}$
  $\overline{a} \cdot \overline{b} = \overline{ab}$

In 3450/4460 you show these are well-defined operations

<u>Ex:</u> $\mathbb{Z}_6 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}\}$

$\overline{3} + \overline{5} = \overline{3+5} = \overline{8} = \overline{2}$

way 1:

$8 \equiv 2 \pmod{6}$
$6 \mid (8-2)$

$\overline{3} + \overline{5} = \overline{8} = \overline{2}$

$n = 6$

$$\begin{array}{r} 1 \\ 6 \,\big)\, 8 \\ -6 \\ \hline 2 \end{array}$$

why works

$8 = 6 \cdot 1 + 2$

$8 - 2 = 6 \cdot 1$

$6 \mid (8-2)$

way 2:

$\overline{5} \cdot \overline{4} = \overline{5 \cdot 4} = \overline{20} = \overline{14} = \overline{8} = \overline{2}$

subtract $n = 6$

$\overline{5} \cdot \overline{4} = \overline{20} = \overline{2}$

$n = 6$

$$\begin{array}{r} 3 \\ 6 \,\big)\, 20 \\ -18 \\ \hline 2 \end{array}$$

$\mathbb{Z}_6$ is a group under $+$.

The identity element is $e = \overline{0}$.

$\overline{y}$ is inverse of $\overline{x}$ if $\overline{x} + \overline{y} = \overline{0}$

$\overline{x}$	inverse of $\overline{x}$
$\overline{0}$	$\overline{0}$
$\overline{1}$	$\overline{5}$
$\overline{2}$	$\overline{4}$
$\overline{3}$	$\overline{3}$
$\overline{4}$	$\overline{2}$
$\overline{5}$	$\overline{1}$

$\overline{0} + \overline{0} = \overline{0}$

$\overline{1} + \overline{5} = \overline{6} = \overline{0}$

$\overline{2} + \overline{4} = \overline{0}$

$\overline{3} + \overline{3} = \overline{0}$

$\overline{4} + \overline{2} = \overline{0}$

$\overline{5} + \overline{1} = \overline{0}$

Theorem: The set $\mathbb{Z}_n$ is a group under addition. The identity element is $e = \overline{0}$. Further, $\mathbb{Z}_n$ is abelian.

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proof: See online notes ∎

Def: A **group table** is a table of all the group calculations, done like this:

G	$\cdots$	b
$\vdots$		
a		$a * b$

Ex: Let's calculate the group table for $\mathbb{Z}_3 = \{\overline{0}, \overline{1}, \overline{2}\}$ under addition.

$\mathbb{Z}_3$	$\overline{0}$	$\overline{1}$	$\overline{2}$
$\overline{0}$	$\overline{0}$	$\overline{1}$	$\overline{2}$
$\overline{1}$	$\overline{1}$	$\overline{2}$	$\overline{0}$
$\overline{2}$	$\overline{2}$	$\overline{0}$	$\overline{1}$

$\boxed{\begin{aligned}\overline{1}+\overline{2} &= \overline{3} \\ &= \overline{0}\end{aligned}}$

$\boxed{\begin{aligned}\overline{2}+\overline{2} &= \overline{4} \\ &= \overline{1}\end{aligned}}$

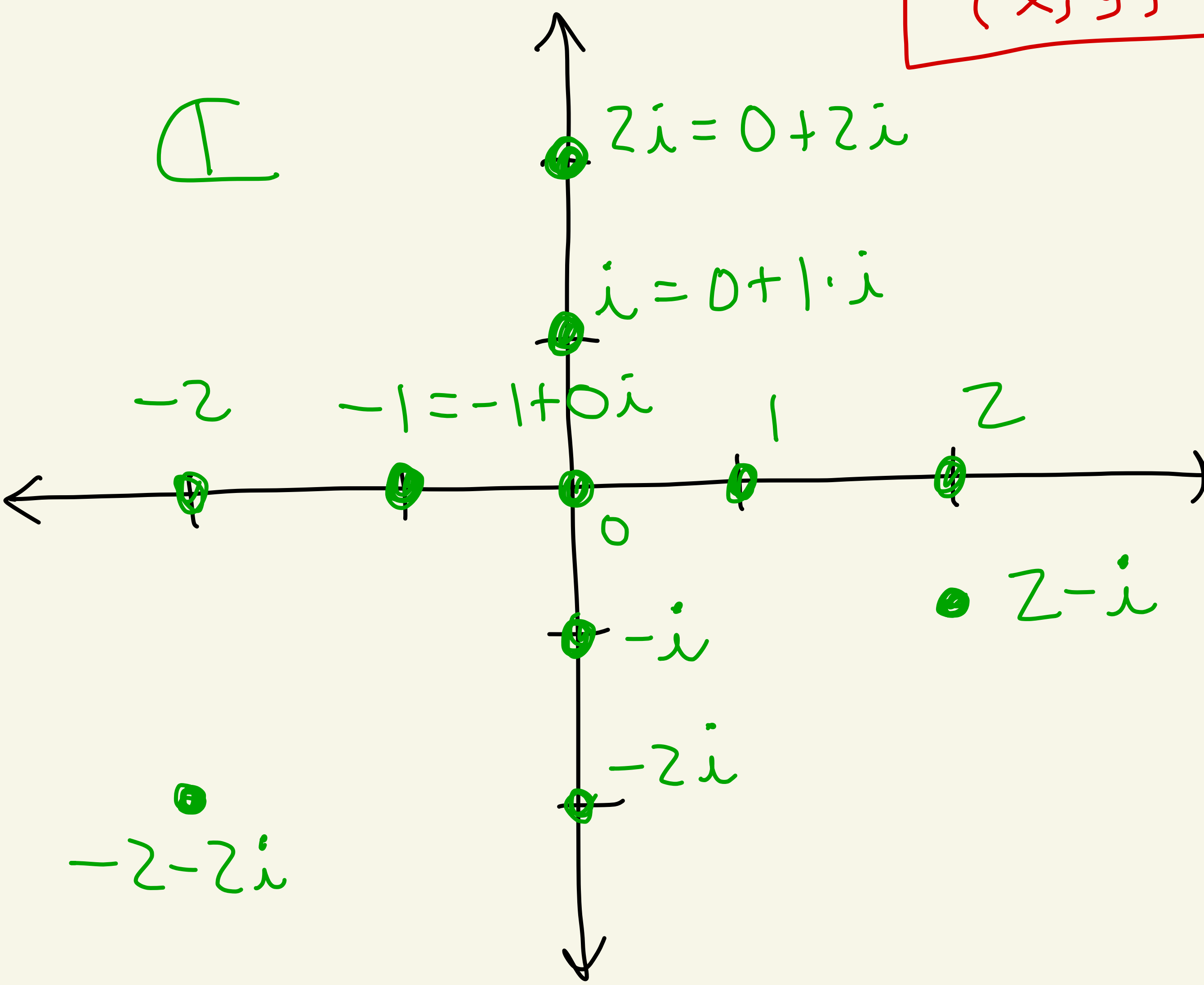
Ex: (the n-th roots of unity) $\boxed{1}$

Recall the set of complex numbers

$$\mathbb{C} = \{ x + iy \mid x, y \in \mathbb{R} \}$$

and $i^2 = -1$.

$x+iy$ is plotted as (x,y)

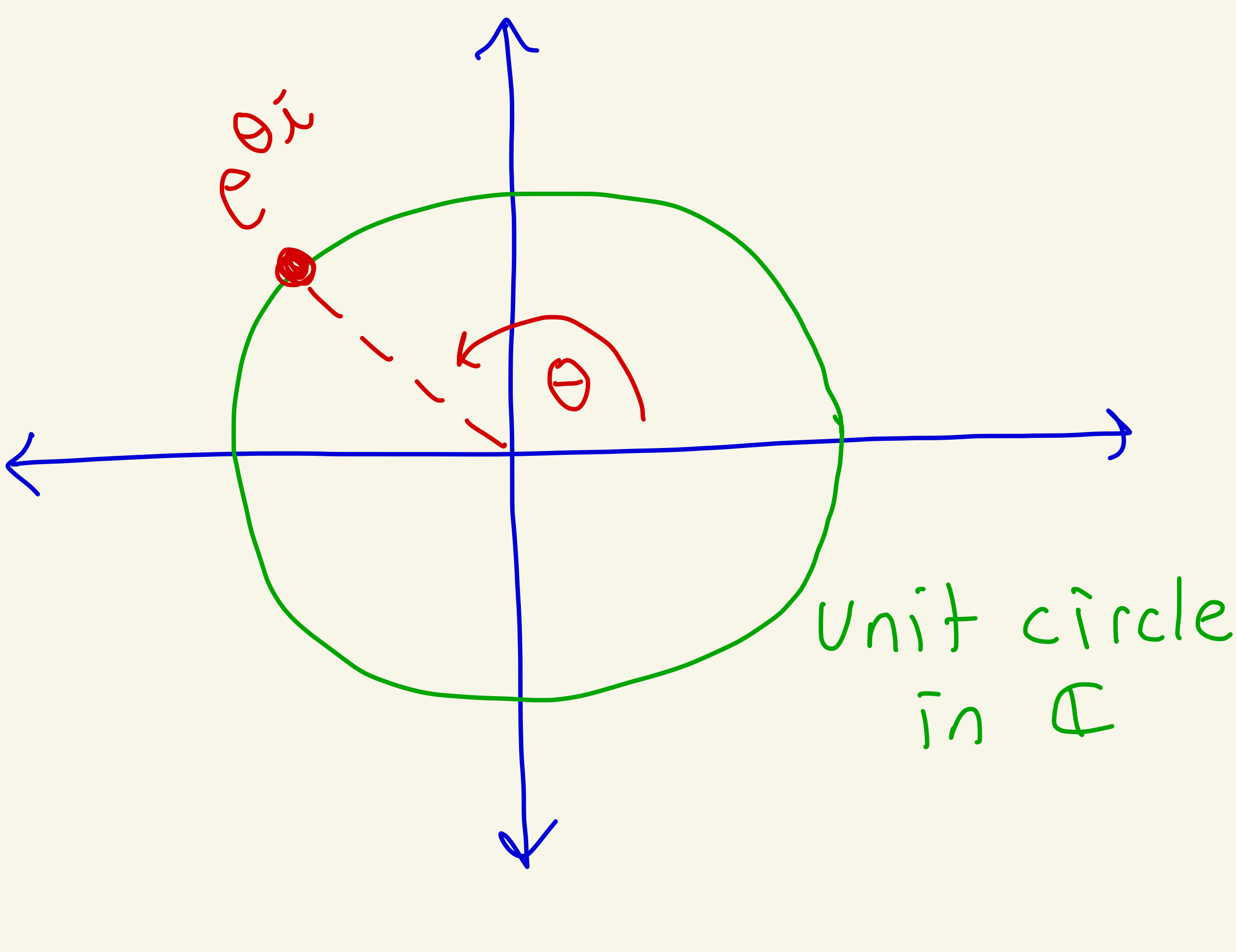


In Math 4680, you will study the complex exponential function. Part of that will involve Euler's formula.

Define

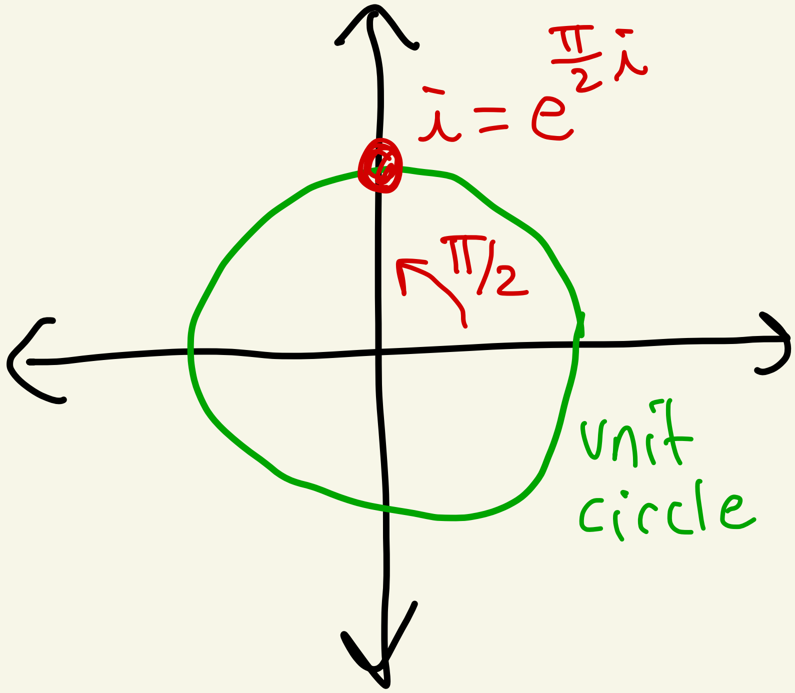
$$e^{\theta i} = \cos(\theta) + i\sin(\theta)$$

where θ is a real number.



Ex:

$$e^{\frac{\pi}{2}i} = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)$$
$$= 0 + 1 \cdot i$$
$$= i$$



$$e^{-\frac{\pi}{3}i} = \cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)$$
$$= \frac{1}{2} - \frac{\sqrt{3}}{2}i$$