Math 4550

12/3/25

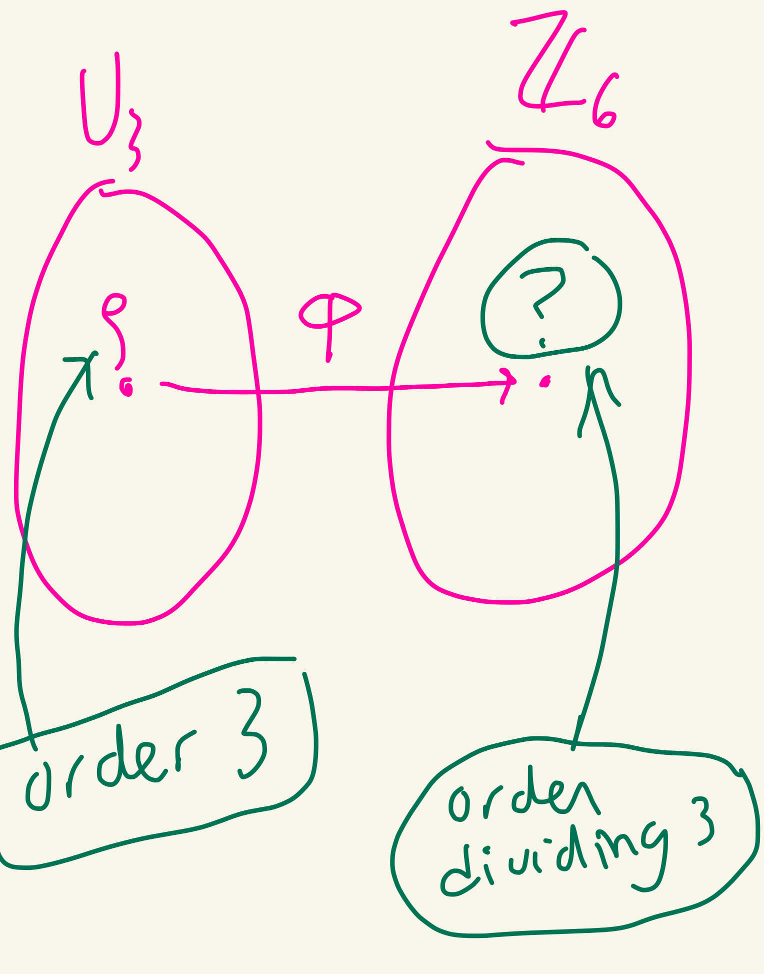
Ex: Find all homomorphisms $\varphi : U_3 \to \mathbb{Z}_6$.

$U_3 = \{1, \rho, \rho^2\}$ where $\rho = e^{2\pi i/3}$

$\rho^3 = 1$

generator order 3

$\mathbb{Z}_6$	order
$\bar{0}$	1
$\bar{1}$	6
$\bar{2}$	3
$\bar{3}$	2
$\bar{4}$	3
$\bar{5}$	6

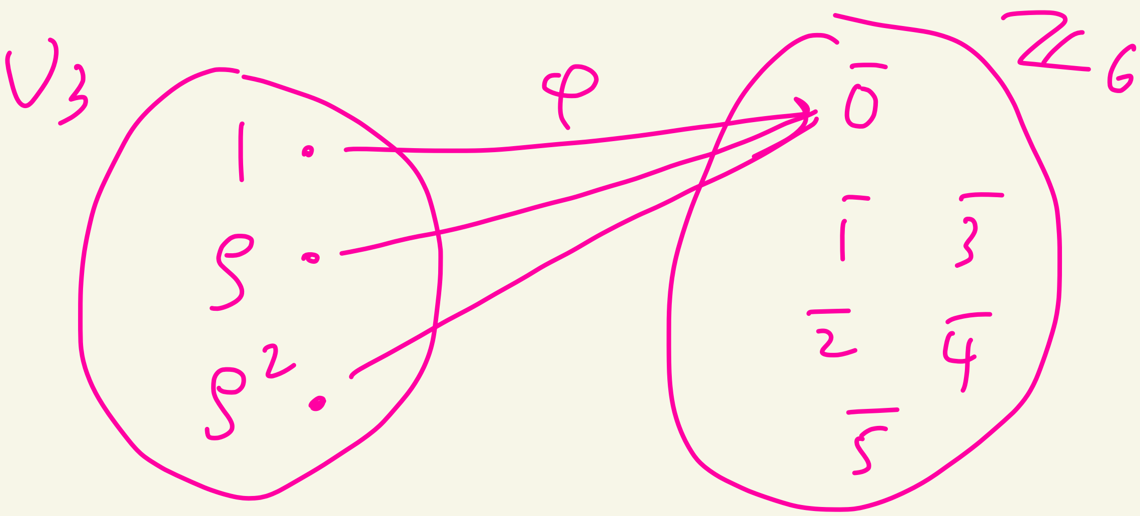


Case 1: $\varphi(\rho) = \bar{0}$

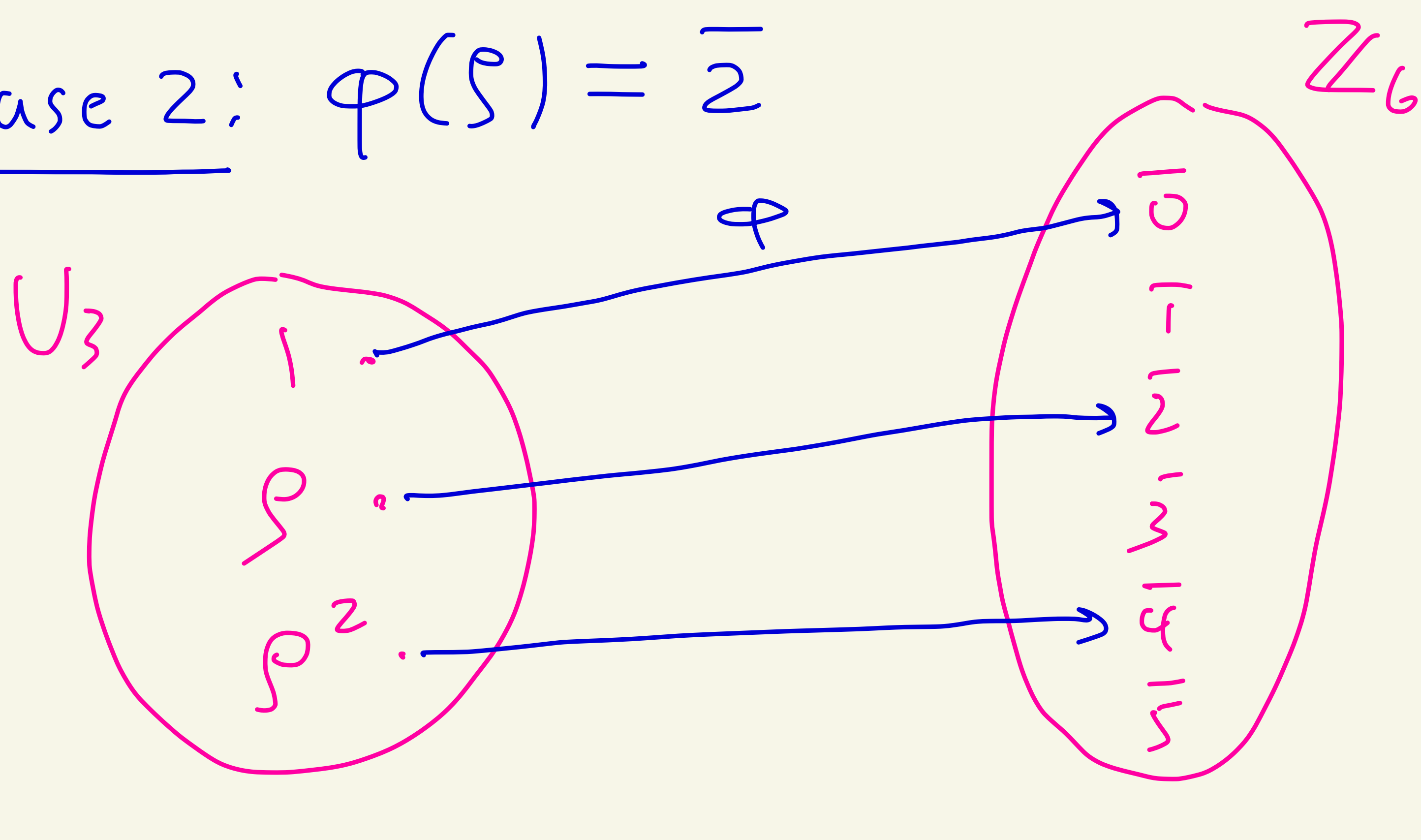
Then, $\varphi(\rho^2) = \varphi(\rho \cdot \rho) = \varphi(\rho) + \varphi(\rho)$

$= \bar{0} + \bar{0} = \bar{0}$

Also, $\varphi(1) = \bar{0}$

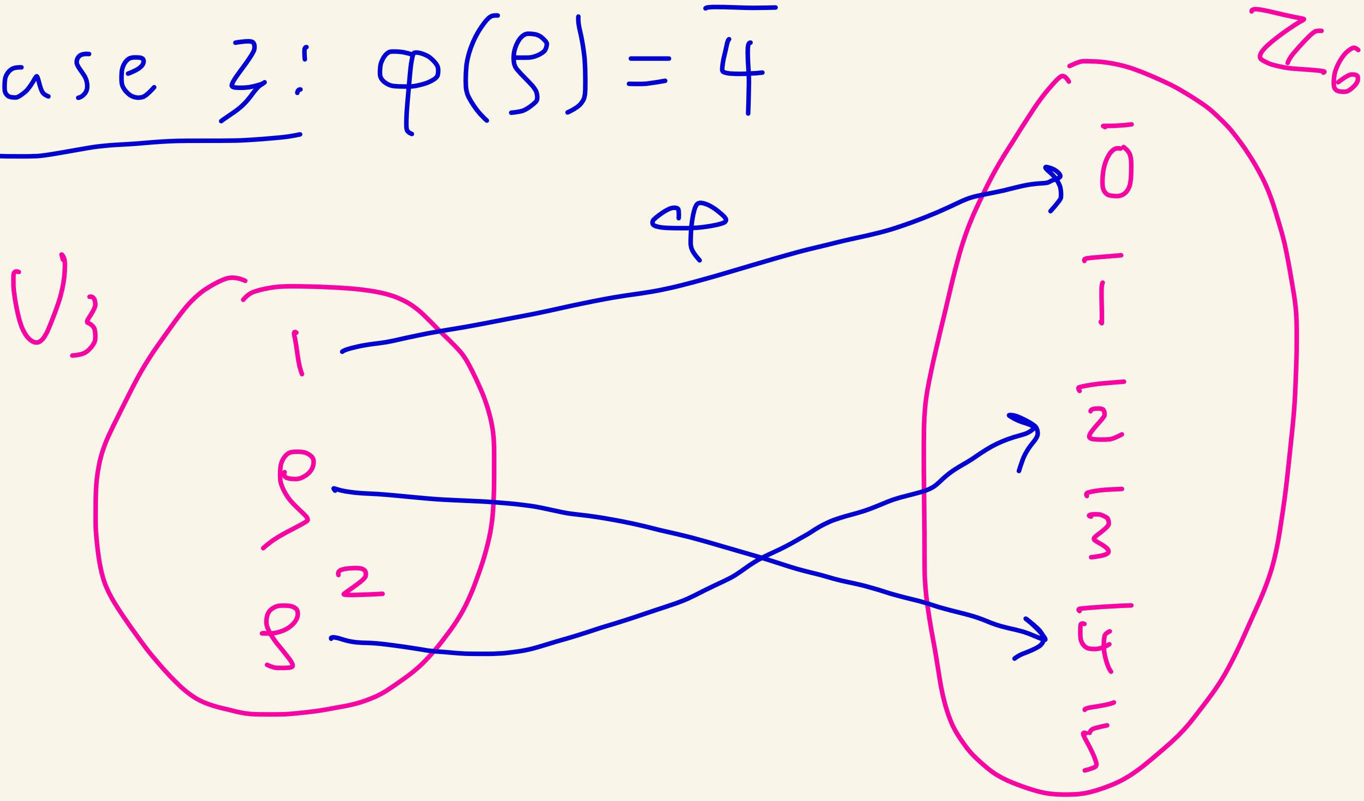


Case 2: $\varphi(\rho) = \bar{2}$



$$\varphi(\rho^2) = \varphi(\rho\cdot\rho) = \varphi(\rho) + \varphi(\rho)$$
$$= \overline{2} + \overline{2} = \overline{4}$$

<u>case 3:</u> $\varphi(\rho) = \overline{4}$



$$\varphi(\rho^2) = \varphi(\rho\cdot\rho) = \varphi(\rho) + \varphi(\rho)$$
$$= \overline{4} + \overline{4} = \overline{8} = \overline{2}$$

Test 1 #2

$D_8 = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$

$r^4 = 1$, $s^2 = 1$, $r^k s = s r^{-k}$
$= s r^{4-k}$

(a) $\underline{\underline{r^{-3} s}}\, r^5 s r^4 = \underline{\underline{s r^3}}\, r^5 s r^4 = s r^8 s r^4$

$r^8 = r^4 r^4 = 1 \cdot 1$
$= 1$

$= s \cdot 1 \cdot s \cdot r^4$
$= s^2 \cdot r^4 = 1 \cdot 1 = 1$

(b) order of sr^3

$sr^3 \neq 1$

$(sr^3)(sr^3) = s r^3 s r^3 = s s r^{-3} r^3$
$= s^2 r^0 = 1 \cdot 1 = 1$

So, sr^3 has order 2.

(c) $\langle r^2 \rangle = \{1, r^2\}$

$\boxed{(r^2)^2 = r^4 = 1}$

(d) inverse of r^3 ?

$r^4 = 1$

$r \cdot r^3 = 1$ $\leftarrow$ $\boxed{(r^3)^{-1} = r}$

Test 1

(A) Show $H = \{2x+3y \mid x, y \in \mathbb{Z}\}$ is a subgroup of $\mathbb{Z}$.

- Note that $0 = 2\underbrace{(0)}_{x} + 3\underbrace{(0)}_{y}$

  So, $0 \in H$.

- Let $a, b \in H$.

  Then, $a = 2x_1 + 3y_1$

  and $b = 2x_2 + 3y_2$

  where $x_1, x_2, y_1, y_2 \in \mathbb{Z}$.

  Then,

  $$a + b = 2x_1 + 3y_1 + 2x_2 + 3y_2$$

  $$= 2(x_1 + x_2) + 3(y_1 + y_2)$$

  Since $x_1 + x_2 \in \mathbb{Z}$ and $y_1 + y_2 \in \mathbb{Z}$

We have $a+b \in H$

- Let $c \in H$.

Then, $c = 2x + 3y$

where $x, y \in \mathbb{Z}$.

We have

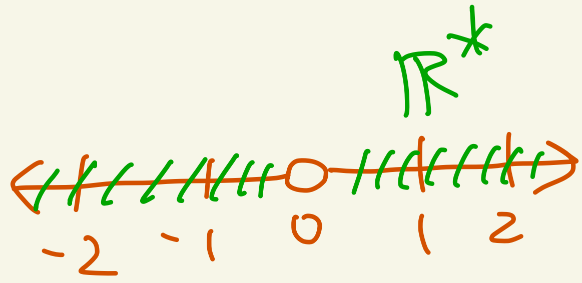
$$-c = -2x - 3y$$
$$= 2(-x) + 3(-y)$$

Since $-x, -y \in \mathbb{Z}$ we know $-c \in H$.

By the above $H \leq \mathbb{Z}$ $\blacksquare$

$\boxed{\text{Test 1}}$

(c) $\mathbb{R}^* = \mathbb{R} - \{0\}$



Define

$$a * b = |ab|$$

For ex: $(-1) * (0.5) = |(-1)(0.5)| = |-0.5| = 0.5$

There is no identity element.

If $e \in \mathbb{R}^*$ was an identity then $e * a = a$ for all $a \in \mathbb{R}^*$

So, $|e \cdot a| = a$ for all $a \in \mathbb{R}^*$

Take $a = -1$, then you'd need $|e \cdot (-1)| = -1$

That's impossible!

$\boxed{\text{HW 6}}$ ④

$$\mathbb{Z}_3 \times \mathbb{Z}_3 = \{ \cancel{(\bar{0},\bar{0})}, \cancel{(\bar{0},\bar{1})}, \cancel{(\bar{0},\bar{2})}, \cancel{(\bar{1},\bar{0})}, \cancel{(\bar{1},\bar{1})}, \cancel{(\bar{1},\bar{2})}, \cancel{(\bar{2},\bar{0})}, \cancel{(\bar{2},\bar{1})}, \cancel{(\bar{2},\bar{2})} \}$$

$$H = \langle (\bar{0},\bar{1}) \rangle$$

$$= \{ (\bar{0},\bar{0}), (\bar{0},\bar{1}), (\bar{0},\bar{2}) \}$$

$$\boxed{(\bar{0},\bar{1}) + (\bar{0},\bar{1})}$$

$$\boxed{(\bar{0},\bar{1}) + (\bar{0},\bar{1}) + (\bar{0},\bar{1}) = (\bar{0},\bar{3}) = (\bar{0},\bar{0})}$$

<u>left cosets:</u>

$$(\bar{0},\bar{0}) + H = \{ (\bar{0},\bar{0}), (\bar{0},\bar{1}), (\bar{0},\bar{2}) \}$$
$$(\bar{1},\bar{0}) + H = \{ (\bar{1},\bar{0}), (\bar{1},\bar{1}), (\bar{1},\bar{2}) \}$$
$$(\bar{2},\bar{0}) + H = \{ (\bar{2},\bar{0}), (\bar{2},\bar{1}), (\bar{2},\bar{2}) \}$$

So,

$$\mathbb{Z}_3 \times \mathbb{Z}_3 / H = \{ \underbrace{(\bar{0},\bar{0}) + H}_{\text{identity}}, (\bar{1},\bar{0}) + H, (\bar{2},\bar{0}) + H \}$$

Note

$$[(\bar{1},\bar{0}) + H] + [(\bar{2},\bar{0}) + H]$$

$$= (\bar{3},\bar{0}) + H = (\bar{0},\bar{0}) + H$$

So, $[(\bar{1},\bar{0}) + H]^{-1} = (\bar{2},\bar{0}) + H$.

Show $\mathbb{Z}_3 \times \mathbb{Z}_3 / H$ is cyclic:

Let's show $(\bar{1},\bar{0}) + H$ is a generator.

$$(\bar{1},\bar{0}) + H \neq (\bar{0},\bar{0}) + H$$

$$[(\bar{1},\bar{0}) + H] + [(\bar{1},\bar{0}) + H] = (\bar{2},\bar{0}) + H$$

$$[(\bar{1},\bar{0}) + H] + [(\bar{1},\bar{0}) + H] + [(\bar{1},\bar{0}) + H] = (\bar{3},\bar{0}) + H$$

$$= (\bar{0},\bar{0}) + H$$