Math 4550

12/1/25

- Today I handed back the tests.
- I will post the solutions later tonight on the website.

- We also covered topic 8 which is not on any test. See the online notes if interested.

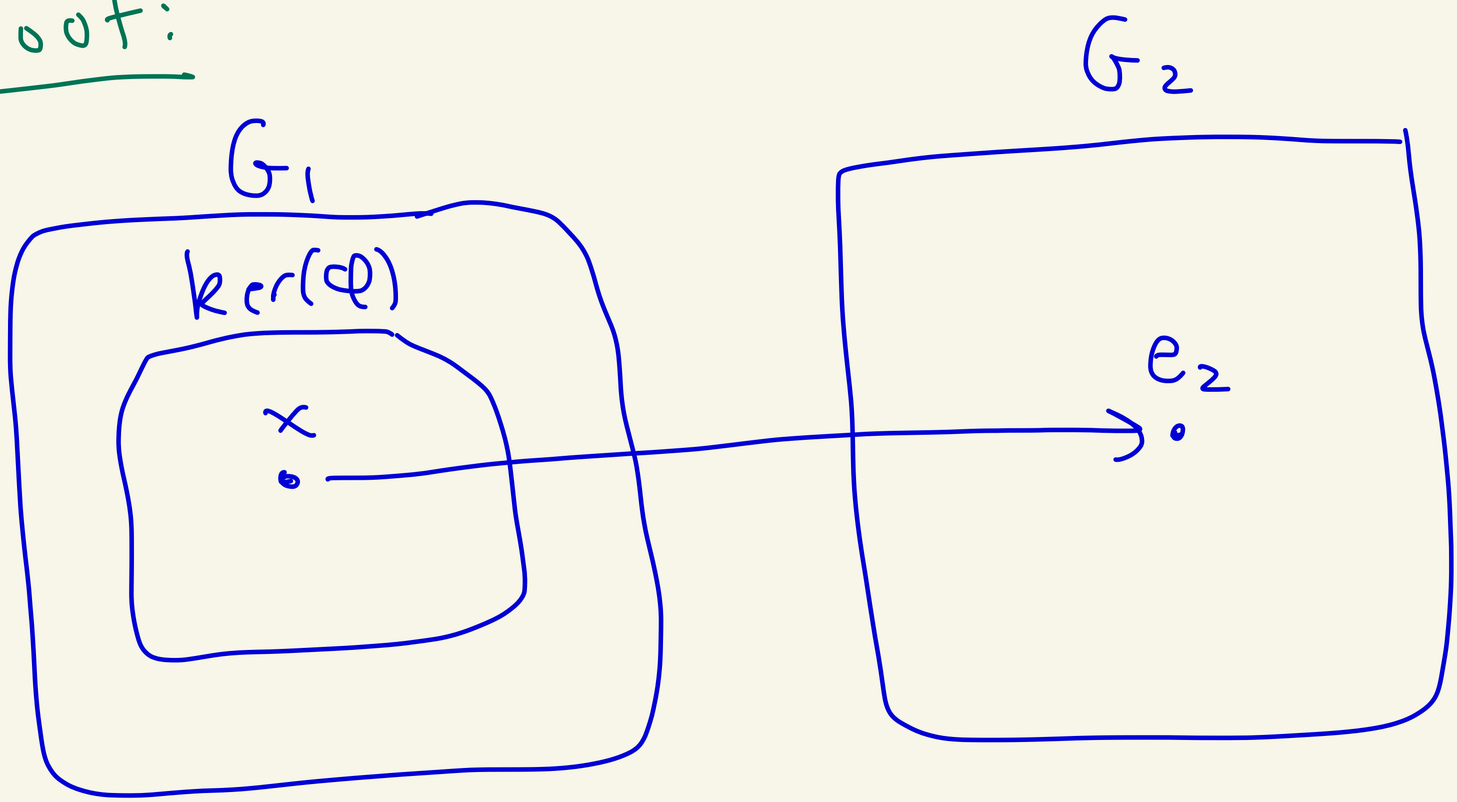
- Weds 12/3 we will review for the final

- Final: Wed, 12/10
  2:30 - 4:30

- Final study guide
  - Review the problems/solutions on test 1 and test 2.
  - Review the problems on the study guides
  - For more practice review HW

A) Let G_1 and G_2 be groups.
Let $\varphi : G_1 \to G_2$ be a homomorphism.
Show that $\ker(\varphi) \leq G_1$.

proof:



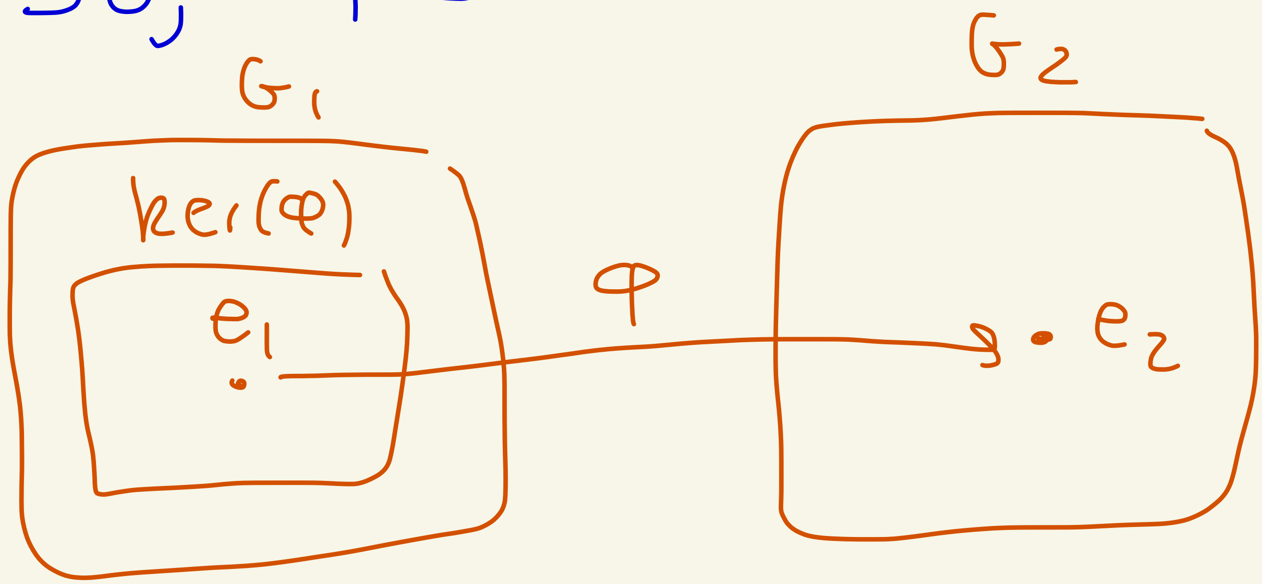
Recall

$$\ker(\varphi) = \{ x \mid \varphi(x) = e_2 \}$$

- Let $e_1 \in G_1$ be the identity of G_1.

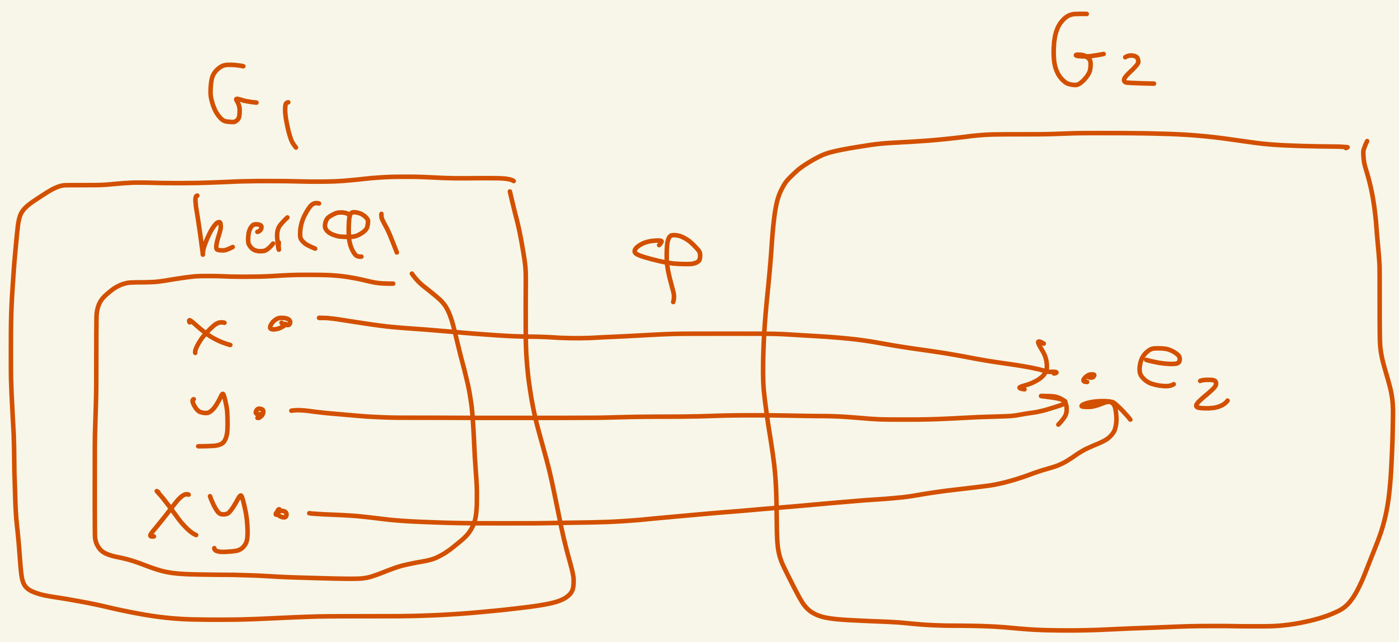
We know from class that $\varphi(e_1) = e_2$.

So, $e_1 \in \ker(\varphi)$



- Let $x, y \in \ker(\varphi)$.

We must show that $xy \in \ker(\varphi)$



Since $x, y \in \ker(\varphi)$ we know $\varphi(x) = e_2$ and $\varphi(y) = e_2$.

Thus,

φ is a hom.

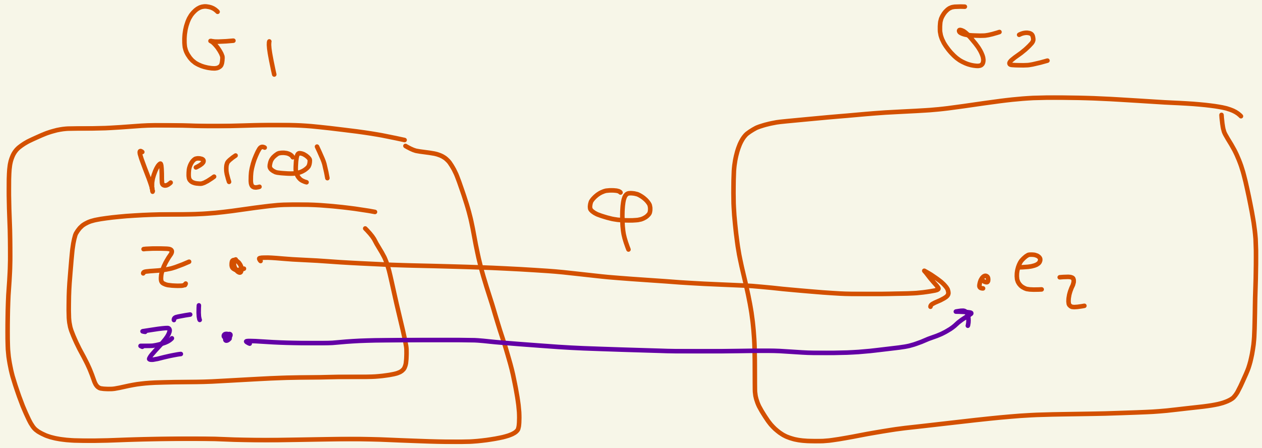
$$\varphi(xy) = \varphi(x)\varphi(y)$$
$$= e_2 e_2$$
$$= e_2$$

So, $xy \in \ker(\varphi)$

- Let $z \in \ker(\varphi)$

  We must show that $z^{-1} \in \ker(\varphi)$.

  Since $z \in \ker(\varphi)$ we know $\varphi(z) = e_2$.



Then,

$$\varphi(z^{-1}) \underset{\varphi \text{ is a hom.}}{=} [\varphi(z)]^{-1} = e_2^{-1} = e_2$$

So, $z^{-1} \in \ker(\varphi)$ $\blacksquare$

HW 4 #6(c)

(B) HW 6 #7

Let G be a group with $|G| = n$.

Prove: If $x \in G$ then $x^n = e$

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proof:

Let $H = \langle x \rangle = \{e, x, x^2, \ldots, x^{k-1}\}$ where k is the order of x.

Lagrange's theorem says that $|H|$ divides $|G|$.

So, k divides n.

Thus, $n = kl$ where $l \in \mathbb{Z}$.

Then,

$$x^n = x^{kl} = (x^k)^l = e^l = e$$

$\blacksquare$

(c) HW 6 #8(d)

Suppose G is a group and $H \trianglelefteq G$.
If G is cyclic, then G/H is cyclic.

<u>proof:</u>

Suppose G is cyclic.
Then there exists $x \in G$ where $G = \langle x \rangle$

$$= \{\ldots, x^{-3}, x^{-2}, x^{-1}, e, x, x^2, x^3, \ldots\}$$

Let's show that
$G/H = \langle xH \rangle$.

says: xH generates G/H

Pick any coset gH from G/H
where $g \in G$.

Since $g \in G$ we know
that $g = x^k$ for some $k \in \mathbb{Z}$.

Thus,

$$gH = x^k H$$
$$= (xH)^k$$

If $k=3$
$$gH = x^3 H$$
$$= (xH)(xH)(xH)$$
$$= (xH)^3$$

So, $G/H = \langle xH \rangle$

$\blacksquare$

In G/H we have: $(aH)(bH) = (ab)H$

Ex:

$$G = D_{12} = \{1, r, r^2, r^3, r^4, r^5, s, sr, sr^2, sr^3, sr^4, sr^5\}$$

$$H = \langle r^3 \rangle = \{1, r^3\} \qquad \boxed{r^6 = 1}$$

$$rH = \{r, r^4\}$$

$$r^2H = \{r^2 \cdot 1, r^2 \cdot r^3\} = \{r^2, r^5\}$$

$$sH = \{s, sr^3\}$$

$$srH = \{sr, sr^4\}$$

$$sr^2H = \{sr^2, sr^5\}$$

$$D_{12}/H = \{H, rH, r^2H, sH, srH, sr^2H\}$$

$$(aH)(bH) = (ab)H \qquad \boxed{r^k s = s r^{-k}}$$

$$(r^2H)(srH) = (\underline{\underline{r^2 s}}\, r)H = (\underline{\underline{s r^{-2}}}\, r)H$$

$$= (sr^{-1})H = (sr^{5})H = (sr^{3})H$$

$$\boxed{r^{-1} = r^{6}r^{-1} = r^{5}}$$

Also:

$$(sr^{-2}r)H = (sr^{6-2}r)H = (sr^{5})H$$

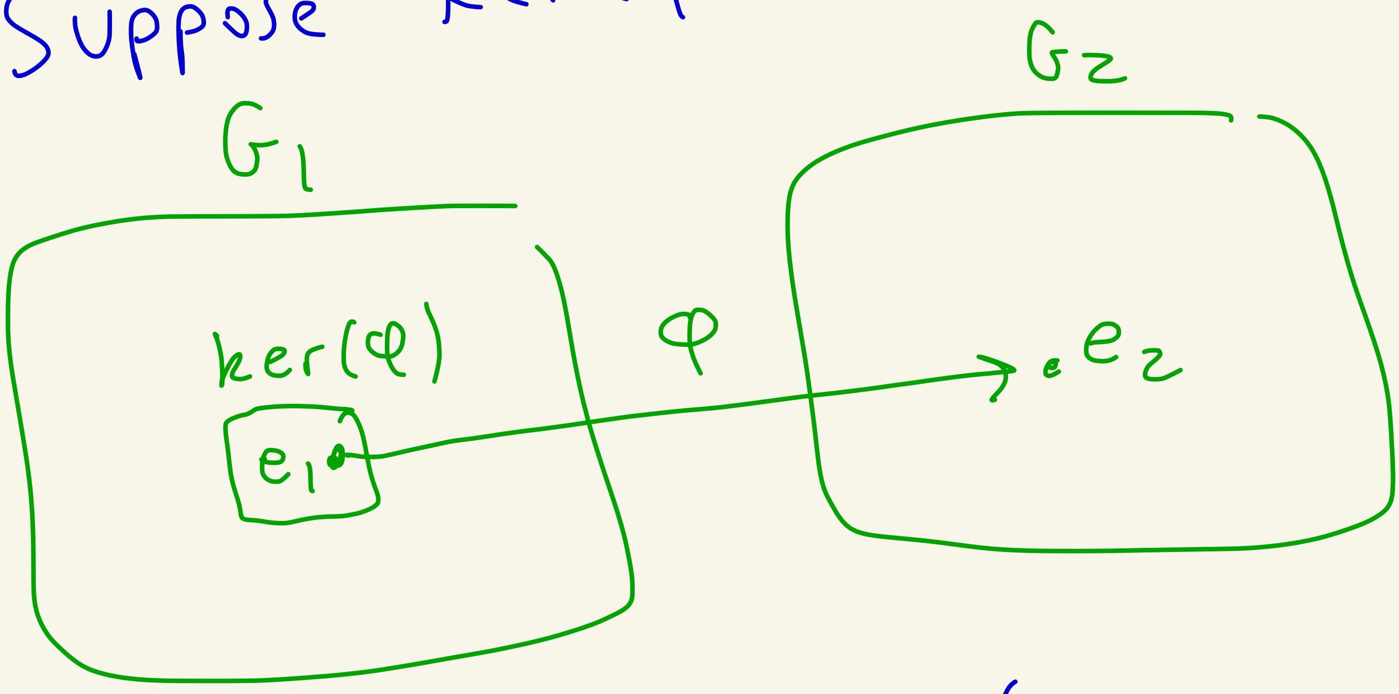
$$\boxed{D_{2n} \leftarrow \begin{array}{l} r^{n} = 1 \\ s^{2} = 1 \\ r^{k}s = sr^{-k} \end{array}}$$

(D) $\varphi : G_1 \to G_2$ hom.

If $\ker(\varphi) = \{e_1\}$,
then φ is 1-1.

<u>proof:</u>

Suppose $\ker(\varphi) = \{e_1\}$.



Suppose $\varphi(x) = \varphi(y)$
where $x, y \in G_1$.

We must show that $x = y$.

Since $\varphi(x) = \varphi(y)$ we have $\varphi(x)\varphi(y)^{-1} = \varphi(y)\varphi(y)^{-1}$

So, $\varphi(x)\varphi(y)^{-1} = e_2$

Thus, $\varphi(x)\varphi(y^{-1}) = e_2$

Thus, $\varphi(xy^{-1}) = e_2$

φ is a hom.

So, $xy^{-1} \in \ker(\varphi)$.

Thus, $xy^{-1} = e_1$.

So, $x = y$. $\blacksquare$

HW 4 #6(e) ($\Leftarrow$)
