Math 4550

10/8/25

HW 3

$\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$, $\mathbb{Z}_3 = \{\bar{0}, \bar{1}, \bar{2}\}$

①

identity

$$\mathbb{Z}_2 \times \mathbb{Z}_3 = \{\boxed{(\bar{0},\bar{0})}, (\bar{0},\bar{1}), (\bar{0},\bar{2}), (\bar{1},\bar{0}), (\bar{1},\bar{1}), (\bar{1},\bar{2})\}$$

$$(\bar{1},\bar{1}) + (\bar{1},\bar{2}) = (\bar{1}+\bar{1}, \bar{1}+\bar{2})$$

$$= (\bar{2}, \bar{3}) = (\bar{0},\bar{0})$$

in $\mathbb{Z}_2$ in $\mathbb{Z}_3$

Thus, $(\bar{1},\bar{1})^{-1} = (\bar{1},\bar{2})$.

What's the inverse of $(\bar{1},\bar{0})$?

$$(\bar{1},\bar{0}) + (\bar{1},\bar{0}) = (\bar{2},\bar{0}) = (\bar{0},\bar{0})$$

in $\mathbb{Z}_2$

So,

$$(\bar{1},\bar{0})^{-1}=(\bar{1},\bar{0}).$$

Find the order of $(\bar{0},\bar{2})$.

$$(\bar{0},\bar{2})\neq(\bar{0},\bar{0})$$

$$(\bar{0},\bar{2})+(\bar{0},\bar{2})=(\bar{0},\bar{4})=(\bar{0},\bar{1})\neq(\bar{0},\bar{0})$$

in $\mathbb{Z}_3$

$$(\bar{0},\bar{2})+(\bar{0},\bar{2})+(\bar{0},\bar{2})=(\bar{0},\bar{6})=(\bar{0},\bar{0})$$

in $\mathbb{Z}_3$

$\boxed{\text{So, } (\bar{0},\bar{2}) \text{ has order } 3}$

Show that $\mathbb{Z}_2 \times \mathbb{Z}_3$ is cyclic by showing that $(\bar{1},\bar{1})$ generates it.

$$\langle(\bar{1},\bar{1})\rangle = \{(\bar{0},\bar{0}), (\bar{1},\bar{1}), \underbrace{(\bar{0},\bar{2})}_{\substack{(\bar{1},\bar{1})+(\bar{1},\bar{1}) \\ =(\bar{2},\bar{2})=(\bar{0},\bar{2})}}, \underbrace{(\bar{1},\bar{0})}_{(\bar{1},\bar{3})}, \underbrace{(\bar{0},\bar{1})}_{(\bar{2},\bar{1})}, (\bar{1},\bar{2})\}$$

$$= \mathbb{Z}_2 \times \mathbb{Z}_3$$

Calculate

$$\langle(\bar{0},\bar{1})\rangle = \{(\bar{0},\bar{0}), (\bar{0},\bar{1}), \overbrace{(\bar{0},\bar{2})}^{(\bar{0},\bar{1})+(\bar{0},\bar{1})}\}$$

order $(\bar{0},\bar{1})$ is 3.

$\boxed{\text{HW 2}}$

4) In $\mathbb{Z}_8$ calculate

$$\langle \overline{4} \rangle = \{ \overline{0}, \overline{4} \}$$

$$\langle \overline{2} \rangle = \{ \overline{0}, \overline{2}, \overline{4}, \overline{6} \}$$

What is the order of $\overline{6}$?

$$\overline{6} \neq \overline{0}$$

$$\overline{6} + \overline{6} = \overline{12} = \overline{4} \neq \overline{0}$$

$$\overline{6} + \overline{6} + \overline{6} = \overline{4} + \overline{6} = \overline{10} = \overline{2} \neq \overline{0}$$

$$\overline{6} + \overline{6} + \overline{6} + \overline{6} = \overline{2} + \overline{6} = \overline{8} = \overline{0}$$

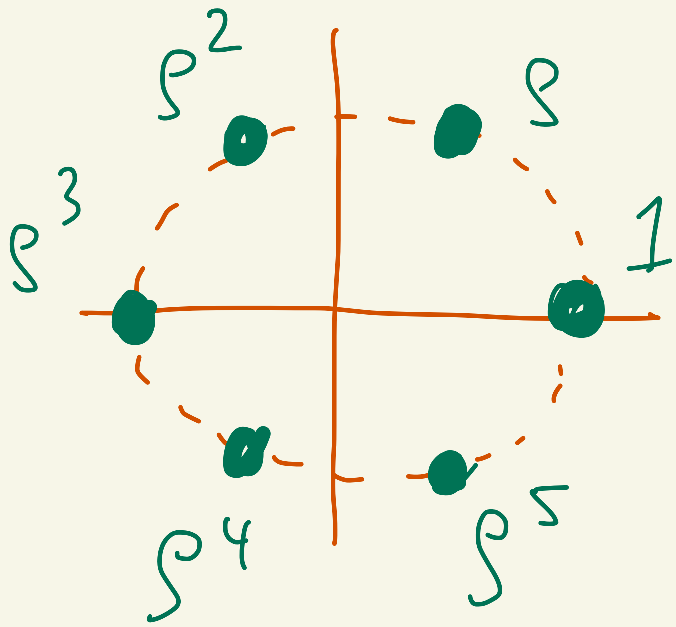
The order of $\bar{6}$ is 4

$\boxed{\text{HW 2}}$

(2) $U_6 = \{1, \rho, \rho^2, \rho^3, \rho^4, \rho^5\}$

$\rho = e^{\frac{2\pi i}{6}}$

$\rho^6 = 1$



Calculate

$\langle \rho^2 \rangle = \{1, \rho^2, \rho^4\}$

$\rho^2 \cdot \rho^2 \cdot \rho^2 = \rho^6 = 1$

$\rho^2 \cdot \rho^2$

The order of ρ^2 is 3.

$\boxed{Q:}$

$D_{10} = \{1, r, r^2, r^3, r^4, s, sr, sr^2, sr^3, sr^4\}$

$r^5 = 1, \quad s^2 = 1,$

$r^k s = sr^{-k} = sr^{5-k}$

Find the order of sr^3.

$sr^3 \neq 1$

$(sr^3)(sr^3) = ssr^{-3}r^3 = s^2 r^0$

$= 1 \cdot 1 = 1$

Thus, sr^3 has order 2.

And

$\langle sr^3 \rangle = \{1, sr^3\}$

$\boxed{\text{HW 2}}$

10 Prove that $H = \{1, r^2, s, sr^2\}$ is a subgroup of $D_8 = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$

H	1	r^2	s	sr^2
1	1	r^2	s	sr^2
r^2	r^2	1	sr^2	s
s	s	sr^2	1	r^2
sr^2	sr^2	s	r^2	1

1 $1 \in H$

2 H closed by table

3 $(r^2)^{-1} = r^2 \in H$

$(s)^{-1} = s \in H$

$(sr^2)^{-1} = sr^2 \in H$

$1^{-1} = 1 \in H$

By ①, ②, ③ we have $H \leq D_8$.

$r^2 r^2 = r^4 = \underline{1}$

$(sr^2)(sr^2) = s s r^{-2} r^2 = s^2 r^0 = 1 \cdot 1 = 1$

$(sr^2)(r^2) = s r^4 = s \cdot 1 = s$

$r^2 s = s r^{-2} = s r^{4-2} = s r^2$

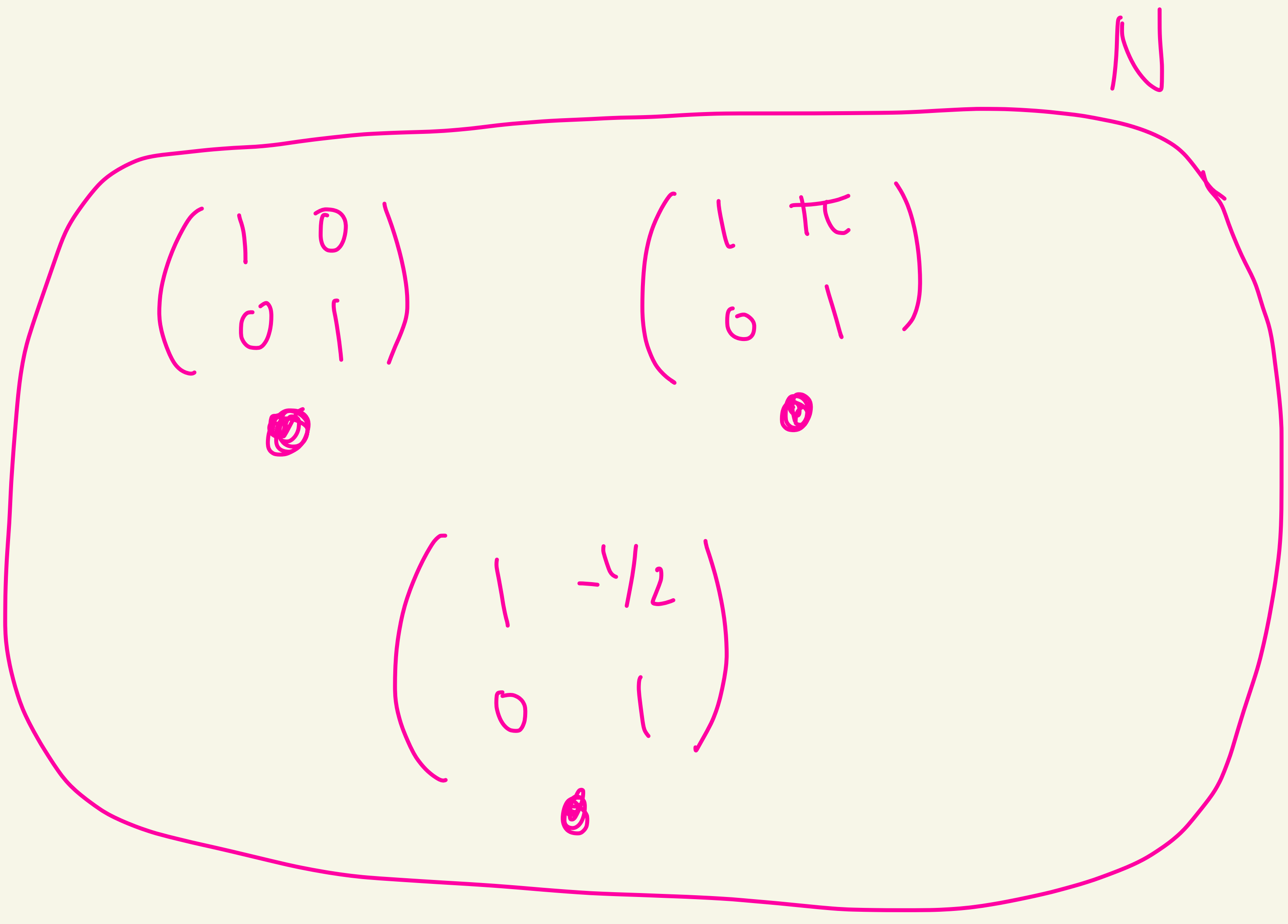
$(sr^2) s = s s r^{-2} = r^{-2} = r^{4-2} = r^2$

$\boxed{\text{HW 2}}$

(11) Show that

$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \,\middle|\, x \in \mathbb{R} \right\}$$

is a subgroup of $GL(2, \mathbb{R})$.



Proof:

① Set $x=0$ we get that $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{identity}}$ is in H.

② Let $A, B \in H$.

Then, $A = \begin{pmatrix} 1 & x_1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & x_2 \\ 0 & 1 \end{pmatrix}$

where $x_1, x_2 \in \mathbb{R}$.

We have

$$AB = \begin{pmatrix} 1 & x_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x_2 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & x_1 + x_2 \\ 0 & 1 \end{pmatrix} \in H$$

because $x_1 + x_2 \in \mathbb{R}$.

③ Let $C \in H$.

Then, $C = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ where $x \in \mathbb{R}$.

And, $C^{-1} = \dfrac{1}{1 \cdot 1 - 0 \cdot x} \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \in H$$

because $-x \in \mathbb{R}$.

By ①, ②, ③, $H \leq GL(2, \mathbb{R})$.

$\blacksquare$

$\boxed{\text{HW 2}}$

(15) Let G be an abelian group. Let H, K be subgroups of G.

Prove

$$HK = \{ hk \mid h \in H \text{ and } k \in K \}$$

is a subgroup of G.

<u>Ex:</u> $G = \mathbb{Z}_{12}$

$H = \langle \bar{3} \rangle = \{ \bar{0}, \bar{3}, \bar{6}, \bar{9} \}$

$K = \langle \bar{4} \rangle = \{ \bar{0}, \bar{4}, \bar{8} \}$

$HK = \{ h + k \mid h \in H, k \in K \}$

$= \{ \underset{\uparrow \, \boxed{h}}{\bar{3}} + \underset{\uparrow \, \boxed{k}}{\bar{4}}, \underset{\uparrow \, \boxed{h}}{\bar{0}} + \underset{\uparrow \, \boxed{k}}{\bar{8}}, \ldots \} = \{ \bar{7}, \bar{8}, \ldots \}$

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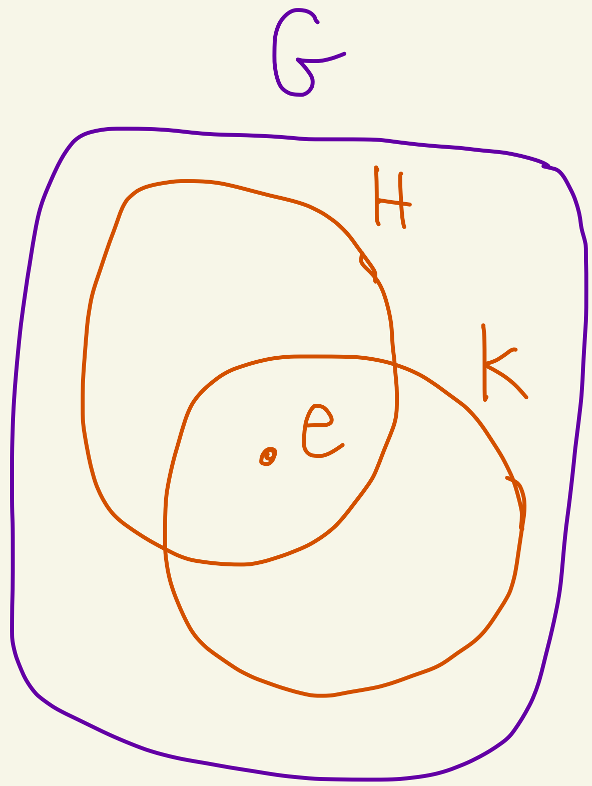
proof:

Let e be the identity of G.

(1) Since $H \le G$ we know $e \in H$.
Since $K \le G$ we know $e \in K$.

So,



$$e = \underbrace{e}_{\text{in } H}\,\underbrace{e}_{\text{in } K} \in HK$$

So, $e \in HK$.

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(2) Let $x, y \in HK$.

We need to show that $xy \in HK$.

Since $x, y \in HK$ we know

$x = h_1 k_1$ and $y = h_2 k_2$

where $h_1, h_2 \in H$ and $k_1, k_2 \in K$.

We have

$$xy = \underbrace{h_1 k_1}_{x} \underbrace{h_2 k_2}_{y} = h_1 h_2 k_1 k_2$$

G is abelian

Since $h_1, h_2 \in H$ and $H \leq G$ we know $h_1 h_2 \in H$.

Since $k_1, k_2 \in K$ and $K \leq G$ we know $k_1 k_2 \in K$.

Thus,

$$xy = (h_1 h_2)(k_1 k_2) \in HK.$$

3) Let $z \in HK$.

We need to show that $z^{-1} \in HK$.

Since $z \in HK$, we have

$z = hk$ where $h \in H$ and $k \in K$.

Since $h \in H$ and $H \leq G$ we know $h^{-1} \in H$.

Since $k \in K$ and $K \leq G$ we know $k^{-1} \in K$.

Then,

$$z^{-1} = (hk)^{-1} = k^{-1}h^{-1} \overset{G \text{ abelian}}{=} h^{-1}k^{-1} \in HK.$$

By 1, 2, 3, $HK \leq G$ ∎