Math 4550

10/6/25

Hw 1

7

$$D_8 = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$$

$$r^4 = 1, \quad s^2 = 1, \quad r^k s = s r^{-k} = s r^{4-k}$$

Calculate

$$(sr^3)(sr^2) = s\,s\,r^{-3}\,r^2$$

$$= s^2 r^{-1} = 1\,r^{4-1}$$

$$= r^3$$

Find the inverse of sr^2.

$$(sr^2)(sr^2) = s\,s\,r^{-2}\,r^2$$

$$= s^2 r^0 = 1 \cdot 1 = 1$$

So, $(sr^2)^{-1} = sr^2$

Find the inverse of r^2.

$$(r^2)(r^2) = r^4 = 1$$

So, $(r^2)^{-1} = r^2$

HW 1

3 $\mathbb{Z}_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$

Find the inverse of $\bar{2}$.

$$\bar{2} + \bar{4} = \bar{6} = \bar{0}$$

So, $\bar{2}^{-1} = \bar{4}$.

Find the inverse of $\bar{5}$.

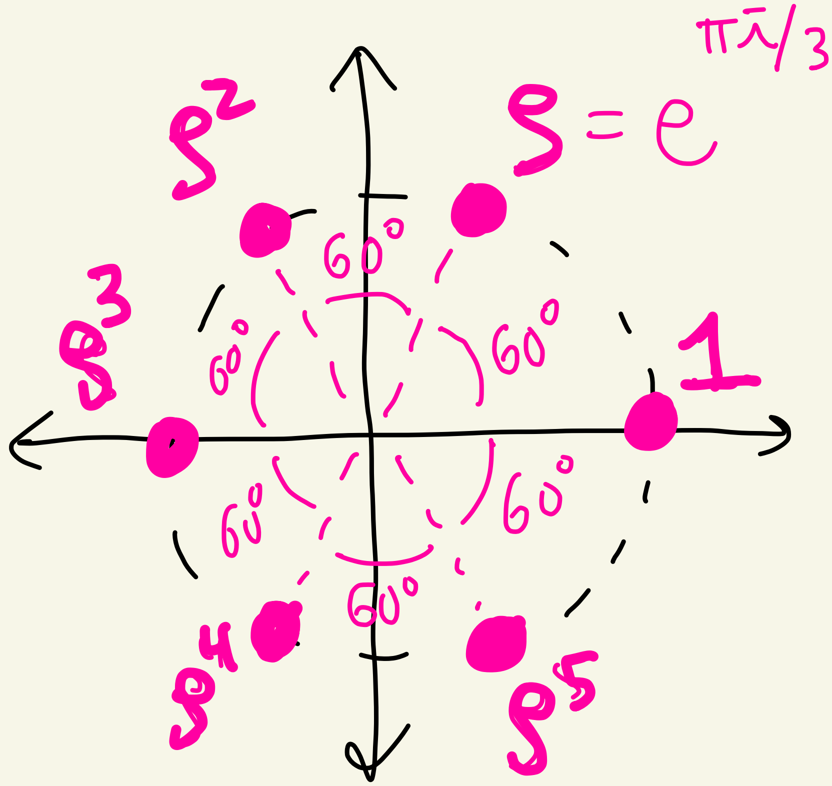
$\bar{1}+\bar{5}=\bar{6}=\bar{0}$

So, $\bar{5}^{-1}=\bar{1}$

$\boxed{\text{HW 1}}$ $\quad \rho^6 = 1$

$U_6 = \{1, \rho, \rho^2, \rho^3, \rho^4, \rho^5\}$

$\rho = e^{2\pi i/6}$



What's the inverse of ρ^2?

$\rho^2 \cdot \rho^4 = \rho^6 = 1$

So, $(\rho^2)^{-1} = \rho^4$.

$\boxed{\text{Hw 1}}$

8 (a) $A = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$ $\quad B = \begin{pmatrix} 1 & 0 \\ 1/2 & 5 \end{pmatrix}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$A^{-1} = \frac{1}{1 \cdot 1 - (-1)(2)} \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$$

$$= \frac{1}{3} \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1/3 & 1/3 \\ -2/3 & 1/3 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/2 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} (1)(1) + (-1)(\frac{1}{2}) & (1)(0) + (-1)(5) \\ (2)(1) + (1)(\frac{1}{2}) & (2)(0) + (1)(5) \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 & -5 \\ 5/2 & 5 \end{pmatrix}$$

$\boxed{\text{HW 1}}$

11 Let G be a group. Suppose every element of G is its own inverse. Prove that G is abelian.

<u>proof:</u>

We know that $x = x^{-1}$ for all $x \in G$.

So, $x^2 = e$ for all $x \in G$.

Let $a, b \in G$.

We want to show $ab = ba$.

We know

$$(ab)^2 = e$$

$$a^2 = e$$

$$b^2 = e$$

So,

$$(ab)(ab) = e$$

Thus,

$$abab = e$$

$$a(abab) = ae$$

$$a^2(bab) = a$$

$$e(bab) = a$$

$$bab = a$$

$$b(bab) = ba$$

$$b^2ab = ba$$

$$eab = ba$$

$$ab = ba$$

So, G is abelian $\blacksquare$

Method 2: Let $a, b \in G$.

Know: $\boxed{\begin{array}{l} a^2b^2 = ee = e \\ (ab)^2 = e \end{array}}$

So, $a^2b^2 = (ab)^2$

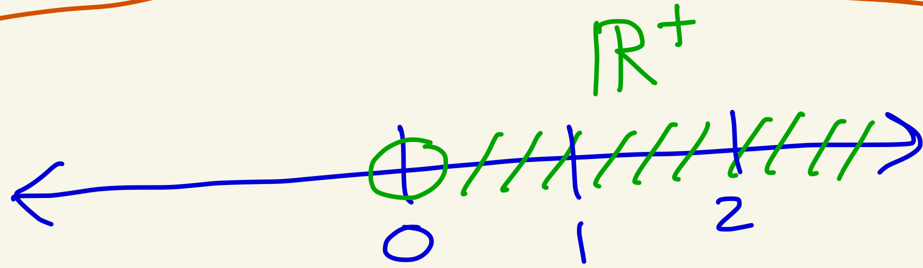
Thus, $aabb = abab$.

So, $a^{-1}(aabb)b^{-1} = a^{-1}(abab)b^{-1}$

Thus, $\underbrace{a^{-1}a}_{e}(ab)\underbrace{bb^{-1}}_{e} = \underbrace{a^{-1}a}_{e}(ba)\underbrace{bb^{-1}}_{e}$

So, $ab = ba$ $\square$

$\boxed{\text{Hw 1}}$



14 $\mathbb{R}^+$ ← positive real numbers

$$a * b = \sqrt{ab}$$

Show its not a group.

<u>Ex calc:</u> $5 * 3 = \sqrt{5 \cdot 3} = \sqrt{15}$

Associativity doesn't work.

For example,

$$1 * (2 * 3) = \sqrt{1 \cdot (2*3)} = \sqrt{\sqrt{2 \cdot 3}} = \sqrt{\sqrt{6}}$$

$$(1*2)*3 = \sqrt{(1*2) \cdot 3} = \sqrt{\sqrt{1 \cdot 2} \cdot 3} = \sqrt{3\sqrt{2}}$$

But $\sqrt{\sqrt{6}} \neq \sqrt{3\sqrt{2}}$

Why?

If $\sqrt{\sqrt{6}} = \sqrt{3\sqrt{2}}$

then $\sqrt{6} = 3\sqrt{2}$,

then $6 = 18$.

square both sides

square both sides

But $6 \neq 18$!

So, $\sqrt{\sqrt{6}} \neq \sqrt{3\sqrt{2}}$.

HW 1

(12) $\mathbb{Z}_{10} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}, \bar{8}, \bar{9}\}$

(a) Show that $\mathbb{Z}_{10}$ is <u>not</u> a group under multiplication.

The identity would have to be $e = \bar{1}$.

$\bar{2}$ has no inverse under mult.

Why?

$\bar{2} \cdot \bar{0} = \bar{0} \neq \bar{1}$

$\bar{2} \cdot \bar{1} = \bar{2} \neq \bar{1}$

$\bar{2} \cdot \bar{2} = \bar{4} \neq \bar{1}$

$\bar{2} \cdot \bar{3} = \bar{6} \neq \bar{1}$

$\bar{2} \cdot \bar{4} = \bar{8} \neq \bar{1}$

$$\bar{2}\cdot\bar{5} = \overline{10} = \bar{0} \neq \bar{1}$$
$$\bar{2}\cdot\bar{6} = \overline{12} = \bar{2} \neq \bar{1}$$
$$\bar{2}\cdot\bar{7} = \overline{14} = \bar{4} \neq \bar{1}$$
$$\bar{2}\cdot\bar{8} = \overline{16} = \bar{6} \neq \bar{1}$$
$$\bar{2}\cdot\bar{9} = \overline{18} = \bar{8} \neq \bar{1}$$

There is no $\bar{x} \in \mathbb{Z}_{10}$ with $\bar{2}\cdot\bar{x} = \bar{1}$.

$\bar{2}$ has no inverse under mult.

$\mathbb{Z}_{10}$ is not a group under mult.

(b) $G = \{\bar{2}, \bar{4}, \bar{6}, \bar{8}\} \subseteq \mathbb{Z}_{10}$

G is a group under mult.

Note: $\bar{a}\cdot(\bar{b}\cdot\bar{c}) = \bar{a}\cdot(\overline{bc})$

$$= \overline{a\cdot(bc)}$$

$$= \overline{(ab)c}$$

$$= \overline{ab} \cdot \overline{c}$$

$$= (\overline{a} \cdot \overline{b}) \cdot \overline{c}$$

So we get associativity.
Let's check the other 3 properties with a table

G	$\overline{2}$	$\overline{4}$	$\overline{6}$	$\overline{8}$
$\overline{2}$	$\overline{4}$	$\overline{8}$	$\overline{2}$	$\overline{6}$
$\overline{4}$	$\overline{8}$	$\overline{6}$	$\overline{4}$	$\overline{2}$
$\overline{6}$	$\overline{2}$	$\overline{4}$	$\overline{6}$	$\overline{8}$
$\overline{8}$	$\overline{6}$	$\overline{2}$	$\overline{8}$	$\overline{4}$

closure	✓
identity	$\overline{6}$
inverses	$\overline{2}^{-1} = \overline{8}$ because $\overline{2} \cdot \overline{8} = \overline{6}$ $\overline{8}^{-1} = \overline{2}$ $\overline{6}^{-1} = \overline{6}$ because $\overline{6} \cdot \overline{6} = \overline{6}$ $\overline{4}^{-1} = \overline{4}$ because $\overline{4} \cdot \overline{4} = \overline{6}$