Math 4550
10/29/25

Theorem: Let H be a subgroup of a group G. Then the following are equivalent.

① H is normal.

② $gHg^{-1} \subseteq H$ for all $g \in G$

③ $gHg^{-1} = H$ for all $g \in G$

where

$$gHg^{-1} = \{ghg^{-1} \mid h \in H\}$$

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proof:

(① $\Rightarrow$ ②)

Assume H is normal.

That is, assume $gH = Hg$ for all $g \in G$.

Fix some $g \in G$.

We will show that $gHg^{-1} \subseteq H$.

Let $y \in gHg^{-1}$.

Then, $y = ghg^{-1}$ where $h \in H$.

We have that $gh \in gH$ and $gH = Hg$,
thus $gh \in Hg$.

Thus, $gh = h_1 g$ where $h_1 \in H$.

So, $y = ghg^{-1} = (gh)g^{-1} = (h_1 g)g^{-1} = h_1 \in H$.

Thus, $gHg^{-1} \subseteq H$.

$\boxed{\textcircled{2} \Rightarrow \textcircled{3}}$

Assume that $xHx^{-1} \subseteq H$ for all $x \in G$.

We must show that $xHx^{-1} = H$ for all $x \in G$.

This comes down to showing that
$H \subseteq xHx^{-1}$ for all $x \in G$.

Let $g \in G$ be fixed.

Let's show $H \subseteq gHg^{-1}$.

Let $h \in H$.

using assumption with $x = g^{-1}$

Then, $g^{-1}hg = g^{-1}h(g^{-1})^{-1} \in H$

Thus,

$$g^{-1}hg = h_2$$

where $h_2 \in H$.

Then, $g(g^{-1}hg)g^{-1} = gh_2g^{-1}$.

So, $h = gh_2g^{-1}$

Hence $h \in gHg^{-1}$.

Thus, $H \subseteq gHg^{-1}$.

[③ ⟹ ①]

Assuming $xHx^{-1} = H$ for all $x \in G$.

We will show that H is normal.

Let $g \in G$.

We will show that $gH = Hg$.

$\boxed{gH \subseteq Hg}$:

Let $y \in gH$.

Then $y = gh$ where $h \in H$.

By assumption $gHg^{-1} = H$. ← using $x = g$

Since $ghg^{-1} \in gHg^{-1}$ we know $ghg^{-1} \in H$.

Thus, $ghg^{-1} = h_3$ where $h_3 \in H$.

So, $gh = h_3 g$.

Thus, $y = gh = h_3 g \in Hg$.

Thus, $gH \subseteq Hg$.

$\boxed{Hg \subseteq gH}$:

Let $z \in Hg$.

Then, $z = h'g$ where $h' \in H$.

By assumption $g^{-1}Hg = H$ ← using $x = g^{-1}$ in assumption $xHx^{-1} = H$

Since $g^{-1}h'g \in g^{-1}Hg$ we know $g^{-1}h'g \in H$.

Thus, $g^{-1}h'g = h_4$ where $h_4 \in H$.

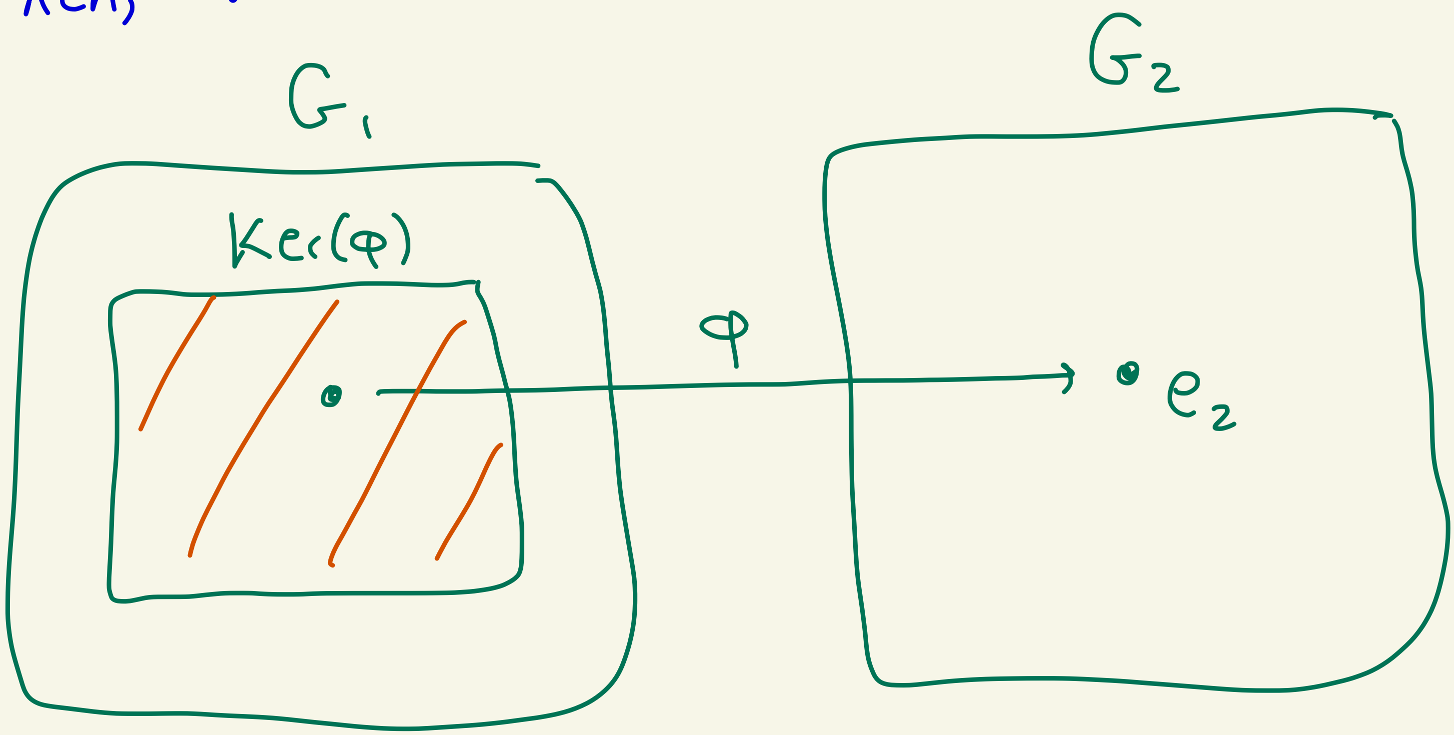
Then, $h'g = gh_4$.

So, $z = h'g = gh_4 \in gH$.

Thus, $Hg \subseteq gH$.

Therefore, $gH = Hg$ and H is normal. $\blacksquare$

Theorem: Let G_1 and G_2 be groups.
Let $\varphi: G_1 \to G_2$ be a homomorphism.
Then, $\ker(\varphi) \trianglelefteq G_1$.



Recall

$$\ker(\varphi) = \{ x \in G_1 \mid \varphi(x) = e_2 \}$$

proof:

Let $H = \ker(\varphi)$.

We previously showed that H is a

subgroup of G_1.

Let's show that H is normal in G_1.

From the previous theorem we will show that $gHg^{-1} \subseteq H$ for all $g \in G_1$

Let $x \in gHg^{-1}$.

Then, $x = ghg^{-1}$ where $h \in H$.

Then,

$$
\begin{aligned}
\varphi(x) &= \varphi(ghg^{-1}) && \text{since } \varphi \text{ is a homomorphism} \\
&= \varphi(g)\varphi(h)\varphi(g^{-1}) && \text{since } h \in H = \ker(\varphi) \\
&= \varphi(g)\, e_2\, \varphi(g^{-1}) \\
&= \varphi(g)\varphi(g^{-1}) && \text{since } \varphi \text{ is a homomorphism} \\
&= \varphi(gg^{-1}) \\
&= \varphi(e_1) && e_1 \text{ identity of } G_1 \\
&= e_2 && \text{fact about homomorphisms}
\end{aligned}
$$

Thus, $x \in \ker(\varphi) = H$.

So, $gHg^{-1} \subseteq H$ no matter what g is.

Thus, $H = \ker(\varphi)$ is normal in G. $\blacksquare$