Math 4550

10/27/25

Ex: $\mathbb{Z}_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$

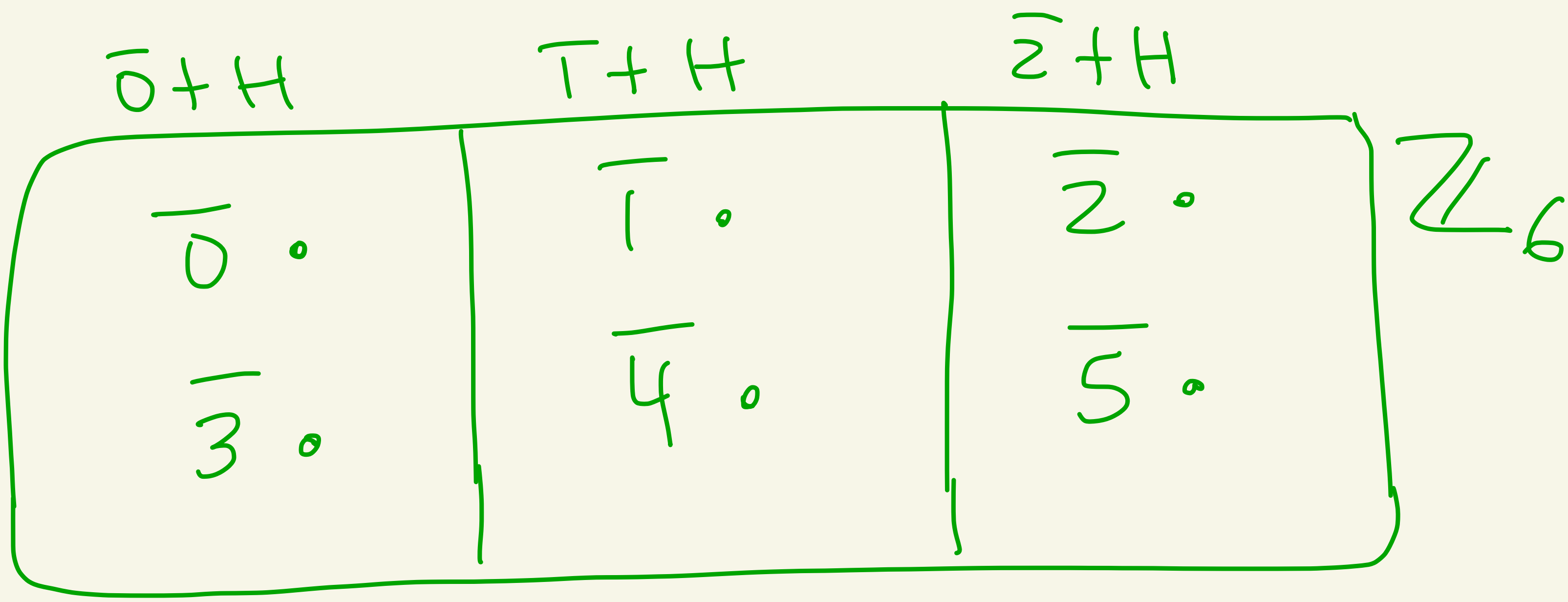
$H = \langle \bar{3} \rangle = \{\bar{0}, \bar{3}\}$

left cosets of H:

$\bar{0} + H = \{\bar{0}, \bar{3}\} = \bar{3} + H$

$\bar{1} + H = \{\bar{1}, \bar{4}\} = \bar{4} + H$

$\bar{2} + H = \{\bar{2}, \bar{5}\} = \bar{5} + H$



set of left cosets

$\mathbb{Z}_6 / H = \{\bar{0}+H, \bar{1}+H, \bar{2}+H\}$

Note

$$6 = |\mathbb{Z}_6| = 2 \cdot 3 = |H| \cdot \left(\begin{matrix}\text{\# of left} \\ \text{cosets}\end{matrix}\right)$$

Lagrange's Theorem

Let G be a finite group and $H \leq G$. Then, $|H|$ divides $|G|$.

proof:

Let $g_1 H, g_2 H, \ldots, g_r H$ be the distinct left cosets that partition G.

Also we know $|H| = |g_i H|$ for all i.

Thus,
$$\begin{aligned} |G| &= |g_1 H| + |g_2 H| + \cdots + |g_r H| \\ &= |H| + |H| + \cdots + |H| \\ &= r|H| \end{aligned}$$

So, $|H|$ divides $|G|$. $\blacksquare$

<u>Corollary:</u> Let G be a group where $|G| = p$ and p is a prime. Then G is cyclic.

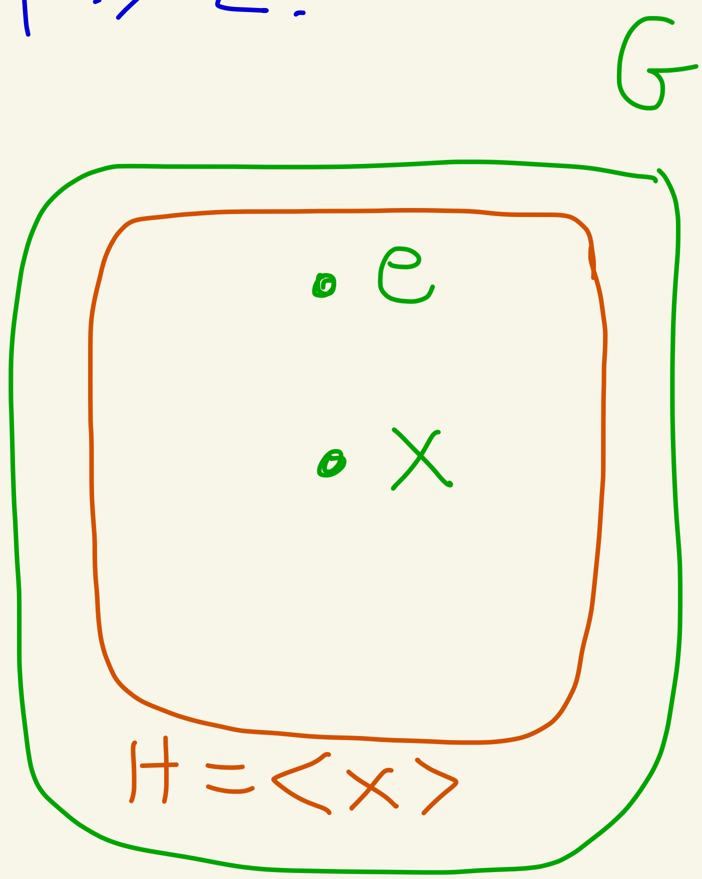
<u>proof:</u>

We have $|G| = p \geq 2$.

Then there exists $x \in G$ with $x \neq e$.

Let $H = \langle x \rangle$.

By Lagrange, $|H|$ divides $|G|$.



So, $|H|$ divides p.

Since p is prime either

$|H| = 1$ or $|H| = p$.

But $|H| \geq 2$ since $e \in H$

and $x \in H$ and $x \neq e$.

So, $|H| = p = |G|$.

Thus, $H = G$.

So, $G = \langle x \rangle$

Thus, G is cyclic. $\blacksquare$

<u>Corollary:</u> Let G be a finite

group. If $x \in G$, then the

order of x divides $|G|$.

Ex: $\mathbb{Z}_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$

$x = \bar{3}$

$\bar{3} + \bar{3} = \bar{0}$

$\bar{3}$ has order 2 which divides $6 = |\mathbb{Z}_6|$

proof: Let $x \in G$ have order n.

Previously we showed that

$$\langle x \rangle = \{e, x, x^2, \ldots, x^{n-1}\}$$

Let $H = \langle x \rangle$.

Then, $|H| = n$.

By Lagrange, $|H|$ divides $|G|$.

So, n divides $|G|$. $\blacksquare$

<u>Def:</u> Let G be a group and $H \leq G$. We say that H is <u>normal</u> in G if $gH = Hg$ for all $g \in G$.

We write $H \trianglelefteq G$ to mean that H is a normal subgroup of G.

<u>Ex:</u> $G = D_6 = \{1, r, r^2, s, sr, sr^2\}$

$H = \langle r \rangle = \{1, r, r^2\}$

Previously, we saw that:

<u>left cosets</u>

$\{1, r, r^2\} = H = rH = r^2H$

$\{s, sr, sr^2\} = sH = srH = sr^2H$

<u>right cosets</u>

$\{1, r, r^2\} = H = Hr = Hr^2$

$\{s, sr, sr^2\} = Hs = Hsr = Hsr^2$

Since $gH = Hg$ for all $g \in D_6$ we have that $H \trianglelefteq D_6$.

Ex: $G = D_6 = \{1, r, r^2, s, sr, sr^2\}$

$H = \langle s \rangle = \{1, s\}$

$$s^2 = 1$$
$$r^3 = 1$$
$$r^k s = s r^{-k}$$

left-cosets:

$H = \{1, s\} = sH$

$rH = \{r \cdot 1, r \cdot s\} = \{r, sr^{-1}\}$

$r^{-1} = r^{3-1} = r^2$

$= \{r, sr^2\}$

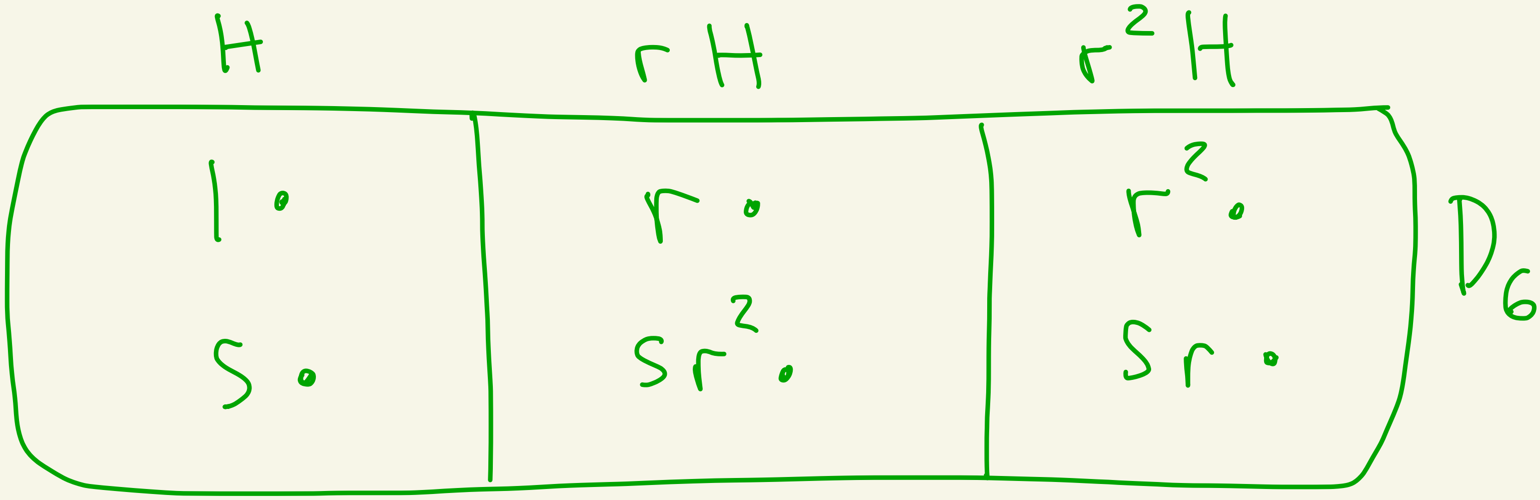
$= sr^2 H$

$r^2 H = \{r^2 \cdot 1, r^2 \cdot s\} = \{r^2, sr^{-2}\}$

$= \{r^2, sr^{3-2}\}$

$= \{r^2, sr\}$

$= srH$

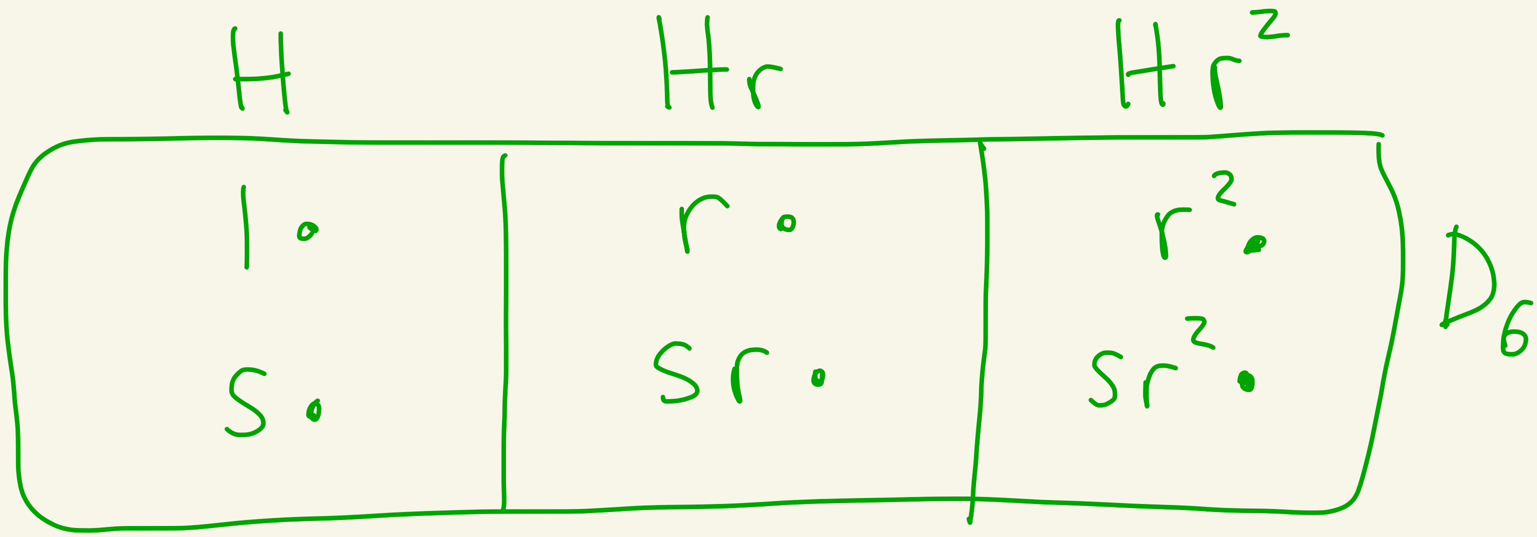


right cosets

$H = \{1, s\} = Hs$

$Hr = \{1 \cdot r, s \cdot r\} = \{r, sr\} = Hsr$

$Hr^2 = \{1 \cdot r^2, s \cdot r^2\} = \{r^2, sr^2\} = Hsr^2$



H is not normal.

For example,

$$rH = \{r, sr^2\} \neq \{r, sr\} = Hr$$

Theorem: Let G be an abelian group and $H \leq G$. Then, $H \trianglelefteq G$.

Proof: Let $g \in G$.

Then,

$$gH = \{gh \mid h \in H\}$$

$$= \{hg \mid h \in H\} = Hg$$

G is abelian

$\blacksquare$

Ex:

$G = \mathbb{Z}_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$ is abelian. So all it's subgroups are normal.

$$\begin{tikzcd}[column sep=small]
& \mathbb{Z}_6 \arrow[dl, dash] \arrow[dr, dash] & \\
\{\bar{0}, \bar{3}\} \arrow[dr, dash] & & \{\bar{0}, \bar{2}, \bar{4}\} \arrow[dl, dash] \\
& \{\bar{0}\} &
\end{tikzcd}$$

Subgroup diagram (topic 5)