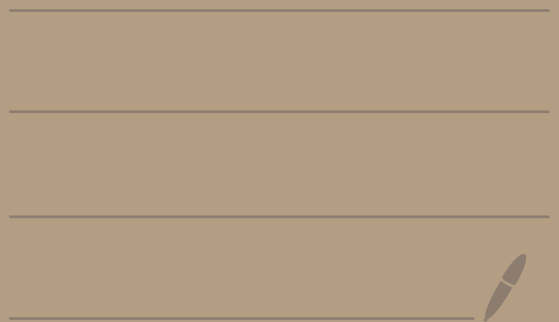


Math 4550

10/22/25



Theorem: Let G be a group and $H \leq G$. Let $a, b \in G$.

Then:

① $a \in aH$

② $aH = bH$ iff $\underbrace{b^{-1}a \in H}_{\text{or } a^{-1}b \in H}$

③ $aH = bH$ iff $\underbrace{a \in bH}_{\text{or } b \in aH}$

④ $aH = bH$ iff $a = bh$ for some $\underbrace{h \in H}$

⑤ The set of left cosets partitions G .

That is,

$$G = \bigcup_{g \in G} gH$$

and given any two left cosets

aH and bH either $aH \cap bH = \emptyset$
or $aH = bH$.

⑥ If H is finite, then
 $|aH| = |H| = |Ha|$

proof: Let e be the identity
of G .

① Since $H \leq G$, we know $e \in H$.
Thus, $a = ae \in aH$.

②

($\Rightarrow$) Suppose $aH = bH$.

By part 1, $a \in aH$.

Since $aH = bH$ we get $a \in bH$.

Thus, $a = bh$ where $h \in H$.

$$\text{So, } b^{-1}a = b^{-1}bh.$$

$$\text{Thus, } b^{-1}a = h.$$

$$\text{So, } b^{-1}a \in H.$$

($\Leftarrow$)

$$\text{Suppose } b^{-1}a \in H.$$

$$\text{Then, } b^{-1}a = h \text{ for some } h \in H.$$

$$\text{So, } \boxed{a = bh}.$$

We need to show that $aH = bH$.

First we show $aH \subseteq bH$.

$$\text{Let } z \in aH.$$

$$\text{Then, } z = ah_1 \text{ where } h_1 \in H.$$

$$\text{So, } z = ah_1 = (bh)h_1 = b(hh_1)$$

Since $H \leq G$ and $h, h_1 \in H$
we know $hh_1 \in H$.

$$\text{So, } z = b(\underbrace{hh_1}_{\text{in } H}) \in bH.$$

Hence $aH \subseteq bH$.

Let's now show $bH \subseteq aH$.

Pick $y \in bH$.

Then, $y = bh_2$ where $h_2 \in H$.

Since $a = bh$ we know
 $b = ah^{-1}$, and also
we know $h^{-1} \in H$.

$$\begin{aligned}
 \text{So, } y &= bh_2 \\
 &= (ah^{-1})h_2 \\
 &= a(\underbrace{h^{-1}h_2}_{\substack{\{h^{-1}, h_2 \in H \rightarrow h^{-1}h_2 \in H \\ \text{because } H \leq G\}}}) \in aH
 \end{aligned}$$

Then, $bH \subseteq aH$.

Since $aH \subseteq bH$ and $bH \subseteq aH$
we have $aH = bH$.

③ HW

④ HW

⑤ Given $g \in G$, we have
 $gH = \{gh \mid h \in H\} \subseteq G$]

So,

$$G = \bigcup_{g \in G} \{g\} \subseteq \bigcup_{g \in G} gH \subseteq G$$

① $\{g\} \subseteq gH$

Ex: $\mathbb{Z}_3 = \{\bar{0}, \bar{1}, \bar{2}\}$

$$\mathbb{Z}_3 = \{\bar{0}\} \cup \{\bar{1}\} \cup \{\bar{2}\}$$

Thus, $G = \bigcup_{g \in G} gH.$

Let $a, b \in G.$

Let's show either

$$aH = bH \text{ or } \underline{aH \cap bH = \emptyset.}$$

disjoint

Assume $aH \cap bH \neq \emptyset.$

Let's show this implies $aH = bH.$

Since $aH \cap bH \neq \emptyset$, there exists $z \in aH \cap bH$.

So, $z \in aH$ and $z \in bH$.

Then, $z = ah_1$ and $z = bh_2$
where $h_1, h_2 \in H$.

Let's show $aH = bH$.

$aH \subseteq bH$:

Let $y \in aH$.

Then, $y = ah$ for some $h \in H$.

We know $ah_1 = z = bh_2$.

So, $y = ah$
 $\quad = (bh_2h_1^{-1})h$

$$ah_1 = bh_2$$

$$a = bh_2h_1^{-1}$$

$$= b(h_2h_1^{-1}h) \in bH$$

in H
because $h_2, h_1^{-1}, h \in H$

Thus, $aH \subseteq bH$.

$$\underline{bH \subseteq aH:}$$

Let $x \in bH$.

Then, $x = bh_3$ where $h_3 \in H$.

Thus,

$$x = bh_3$$

$$= (ah_1h_2^{-1})h_3$$

$$\begin{aligned} ah_1 &= bh_2 \\ ah_1 h_2^{-1} &= b \end{aligned}$$

$$= a(h_1 h_2^{-1} h_3) \in aH$$

in H
because $h_1, h_2^{-1}, h_3 \in H$

So, $bH \subseteq aH$.

Summary: If $aH \cap bH \neq \emptyset$,
then $aH = bH$.

⑥ HW

