Math 4550

10/22/25

Theorem: Let G be a group and $H \leq G$. Let $a, b \in G$.

Then:

① $a \in aH$

② $aH = bH$ iff $\underbrace{b^{-1}a \in H}_{\text{(or } a^{-1}b \in H\text{)}}$

③ $aH = bH$ iff $\underbrace{a \in bH}_{\text{(or } b \in aH\text{)}}$

④ $aH = bH$ iff $a = bh$ for some $h \in H$

⑤ The set of left cosets partitions G.

That is,

$$G = \bigcup_{g \in G} gH$$

and given any two left cosets

aH and bH either $aH \cap bH = \emptyset$ or $aH = bH$.

⑥ If H is finite, then

$$|aH| = |H| = |Ha|$$

<u>proof:</u> Let e be the identity of G.

① Since $H \leq G$, we know $e \in H$.
Thus, $a = ae \in aH$.

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②

($\Rightarrow$) Suppose $aH = bH$.

By part 1, $a \in aH$.

Since $aH = bH$ we get $a \in bH$.

Thus, $a = bh$ where $h \in H$.

So, $b^{-1}a = b^{-1}bh$.

Thus, $b^{-1}a = h$.

So, $b^{-1}a \in H$.

$(\Leftarrow)$

Suppose $b^{-1}a \in H$.

Then, $b^{-1}a = h$ for some $h \in H$.

So, $\boxed{a = bh}$.

We need to show that $aH = bH$.

First we show $aH \subseteq bH$.

Let $z \in aH$.

Then, $z = ah_1$ where $h_1 \in H$.

So, $z = ah_1 = (bh)h_1 = b(hh_1)$

Since $H \leq G$ and $h, h_1 \in H$

we know $hh_1 \in H$.

So, $z = b\underbrace{(hh_1)}_{\text{in } H} \in bH$.

Hence $aH \subseteq bH$.

Let's now show $bH \subseteq aH$.

Pick $y \in bH$.

Then, $y = bh_2$ where $h_2 \in H$.

Since $a = bh$ we know

$b = ah^{-1}$, and also

we know $h^{-1} \in H$.

So, $y = bh_2$

$= (ah^{-1})h_2$

$= a\underbrace{(h^{-1}h_2)}_{\left(h^{-1}, h_2 \in H \to h^{-1}h_2 \in H \text{ because } H \leq G\right)} \in aH$

Then, $bH \subseteq aH$.

Since $aH \subseteq bH$ and $bH \subseteq aH$

we have $aH = bH$.

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③ HW

④ HW

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⑤ Given $g \in G$, we have

$gH = \{gh \mid h \in H\} \subseteq G$

So,

$$G = \bigcup_{g\in G} \{g\} \overset{\text{① } \{g\}\subseteq gH}{\subseteq} \bigcup_{g\in G} gH \subseteq G$$

Ex: $\mathbb{Z}_3 = \{\bar{0}, \bar{1}, \bar{2}\}$

$\mathbb{Z}_3 = \{\bar{0}\} \cup \{\bar{1}\} \cup \{\bar{2}\}$

Thus, $G = \bigcup_{g\in G} gH$.

Let $a, b \in G$.

Let's show either

$aH = bH$ or $\underbrace{aH \cap bH = \emptyset}_{\text{disjoint}}$.

Assume $aH \cap bH \neq \emptyset$.

Let's show this implies $aH = bH$.

Since $aH \cap bH \neq \emptyset$, there exists $z \in aH \cap bH$.

So, $z \in aH$ and $z \in bH$.

Then, $z = ah_1$ and $z = bh_2$ where $h_1, h_2 \in H$.

Let's show $aH = bH$.

<u>$aH \subseteq bH$:</u>

Let $y \in aH$.

Then, $y = ah$ for some $h \in H$.

We know $ah_1 = z = bh_2$.

So, $y = ah$
$= (bh_2h_1^{-1})h$

$$ah_1 = bh_2$$
$$a = bh_2h_1^{-1}$$

$$= b\underbrace{(h_2h_1^{-1}h)}_{\text{in } H} \in bH$$

because $h_2, h_1^{-1}, h \in H$

Thus, $aH \subseteq bH$.

<u>$bH \subseteq aH$:</u>

Let $x \in bH$.

Then, $x = bh_3$ where $h_3 \in H$.

Thus,

$$x = bh_3$$

$$= (ah_1h_2^{-1})h_3$$

$$\boxed{\begin{aligned} ah_1 &= bh_2 \\ ah_1h_2^{-1} &= b \end{aligned}}$$

$$= a\underbrace{(h_1h_2^{-1}h_3)}_{\text{in } H} \in aH$$

because $h_1, h_2^{-1}, h_3 \in H$

So, $bH \subseteq aH$.

Summary: If $aH \cap bH \neq \phi$, then $aH = bH$.

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⑥ HW

$\square$