Math 4550

10/20/25

(Topic 5 continued...)

Theorem: (Classification of cyclic groups)

Let G be a cyclic group.

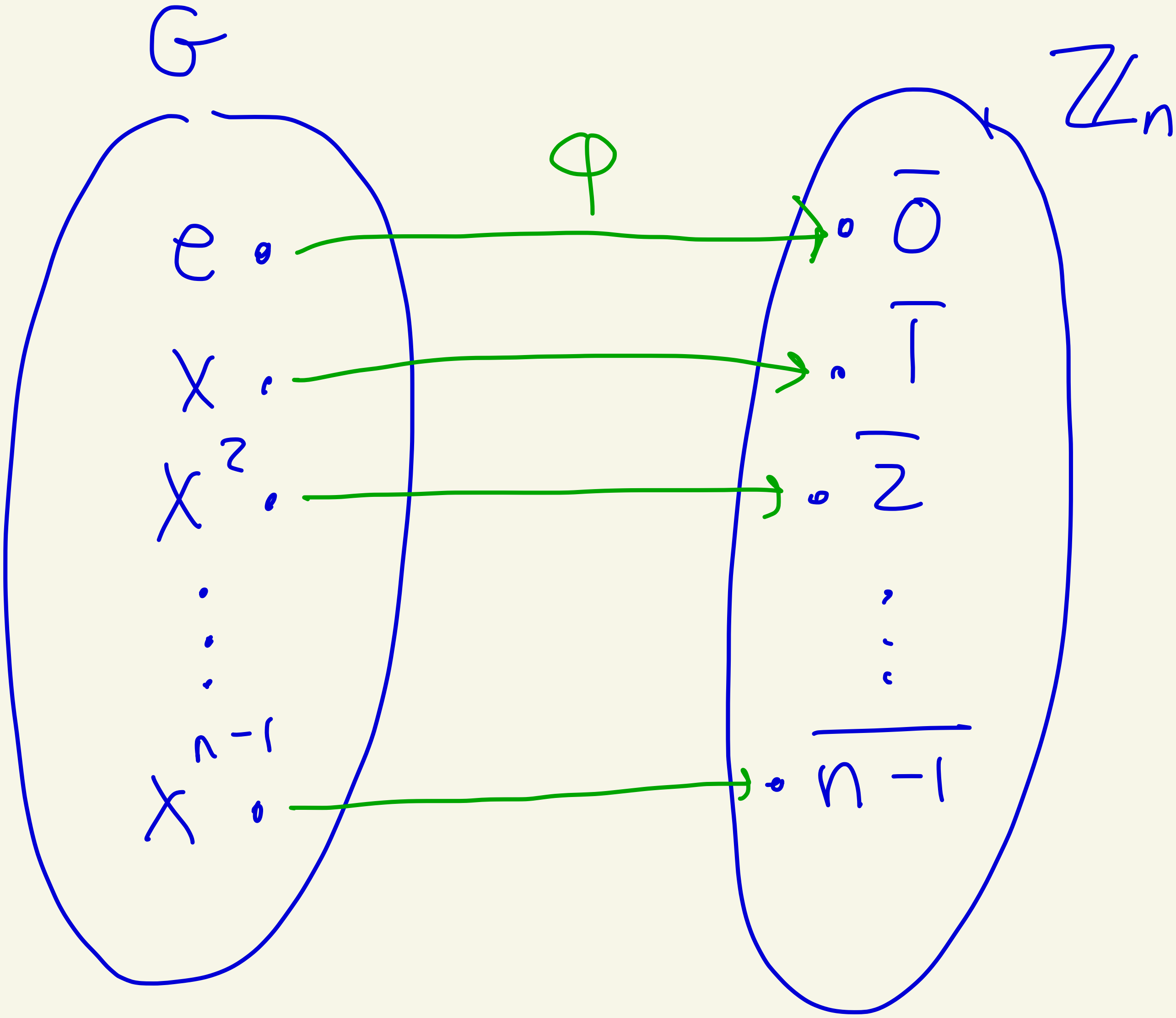
- If $|G| = n$, then $G \cong \mathbb{Z}_n$
- If $|G| = \infty$, then $G \cong \mathbb{Z}$

$\boxed{\text{Proof:}}$ Suppose G is cyclic.

Case 1: Suppose $|G| = n$.

Then there exists $x \in G$ of order n where

$$G = \langle x \rangle = \{e, x, x^2, \ldots, x^{n-1}\}$$



Note that $\bar{1}$ in $\mathbb{Z}_n$ has order n.
By a previous theorem we can set $\varphi(x) = \bar{1}$ because $\bar{1}$ has order n which divides the order of x which is n.

By the previous theorem we can set then

$$\varphi(x^k) = \varphi(x \cdot x \cdots x)$$
$$= \underbrace{\varphi(x) + \varphi(x) + \cdots + \varphi(x)}_{k \text{ times}}$$
$$= \overline{1} + \overline{1} + \cdots + \overline{1}$$
$$= \overline{k}$$

By the same theorem, φ is a homomorphism.

By the picture we can see that φ is 1-1 and onto.

So, φ is an isomorphism.

Thus, $G \cong \mathbb{Z}_n$.

Case 2: Suppose $|G| = \infty$.

Since G is cyclic, there exists $x \in G$ where

$$G = \langle x \rangle = \{ \ldots, x^{-3}, x^{-2}, x^{-1}, e, x, x^2, x^3, \ldots \}$$

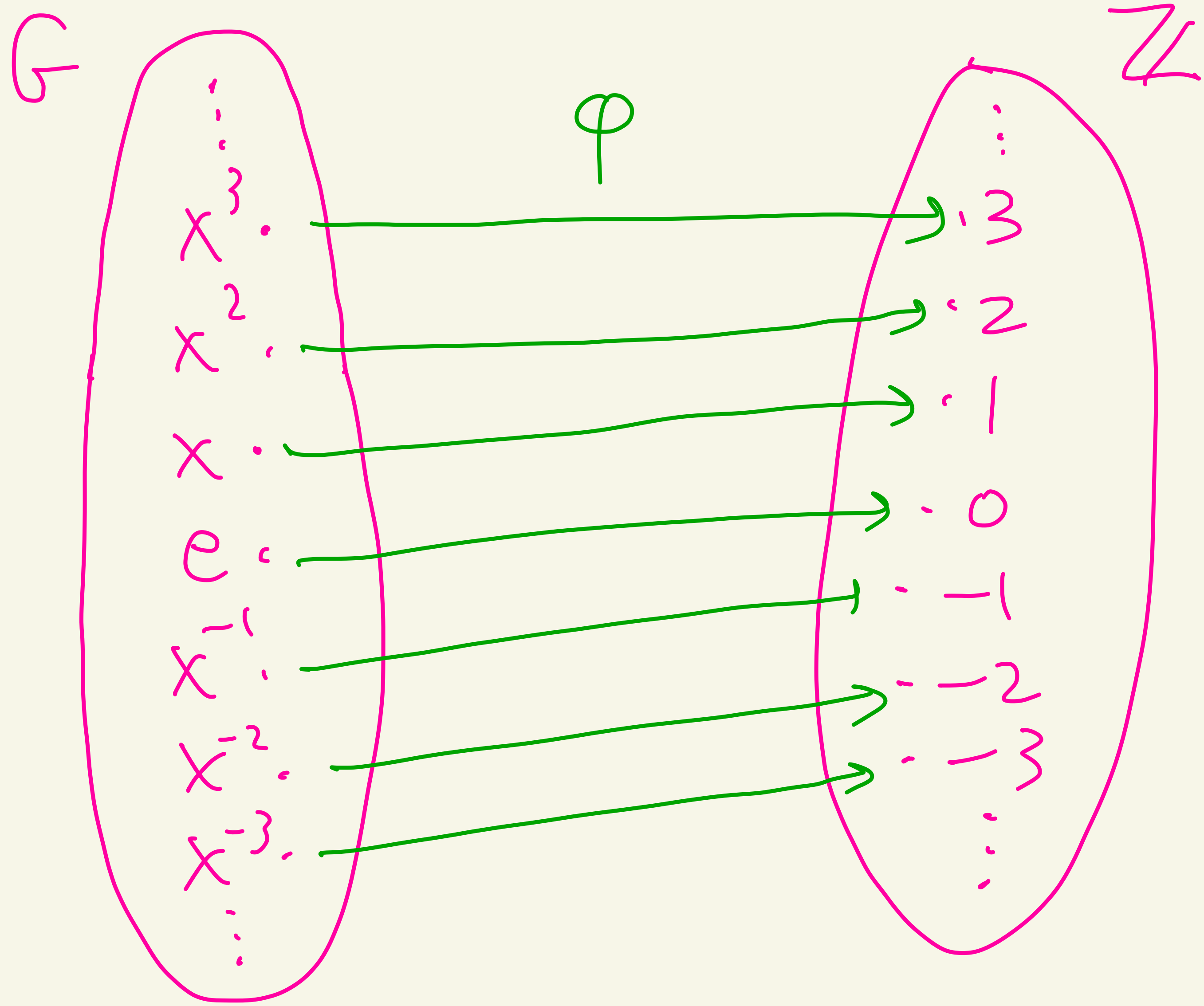
Let's construct a homomorphism $\varphi : G \to \mathbb{Z}$.

Define $\varphi(x) = 1$.

Then the homomorphisms out of cyclic groups theorem tells us that

$$\varphi(x^k) = 1 + 1 + \cdots + 1 = k$$

is a homomorphism.



φ is 1-1 and onto.

So, φ is an isomorphism and $G \cong \mathbb{Z}$. $\blacksquare$

Topic 6 - Normal subgroups and factor groups

<u>Def:</u> Let G be a group and $H \leq G$. Let $g \in G$.

The <u>left coset of H containing g</u> is

$$gH = \{ gh \mid h \in H \}$$

The <u>right coset of H containing g</u> is

$$Hg = \{ hg \mid h \in H \}$$

The set of left cosets is denoted by $\underbrace{G/H}$

read: "G mod H"

Ex:

$$G = D_6 = \{1, r, r^2, s, sr, sr^2\}$$

where $r^3 = 1$, $s^2 = 1$, $r^k s = s r^{-k}$

$$H = \langle r \rangle = \{1, r, r^2\}$$

left-cosets

$$\left.\begin{array}{l}
1 \cdot H = \{1\cdot 1, 1 \cdot r, 1 \cdot r^2\} = \{1, r, r^2\} \\
rH = \{r \cdot 1, r \cdot r, r \cdot r^2\} = \{r, r^2, 1\} \\
r^2 H = \{r^2 \cdot 1, r^2 \cdot r, r^2 \cdot r^2\} = \{r^2, 1, r\}
\end{array}\right\} \text{SAME}$$

$$\left.\begin{array}{l}
sH = \{s \cdot 1, s \cdot r, s \cdot r^2\} = \{s, sr, sr^2\} \\
srH = \{sr \cdot 1, sr \cdot r, sr \cdot r^2\} = \{sr, sr^2, s\} \\
sr^2 H = \{sr^2 \cdot 1, sr^2 \cdot r, sr^2 \cdot r^2\} = \{sr^2, s, sr\}
\end{array}\right\} \text{SAME}$$

There are two left-cosets

$$\{1, r, r^2\} = H = rH = r^2H$$

$$\{s, sr, sr^2\} = sH = srH = sr^2H$$

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right-cosets:

$$H \cdot 1 = \{1 \cdot 1, r \cdot 1, r^2 \cdot 1\} = \{1, r, r^2\}$$

$$H \cdot r = \{1 \cdot r, r \cdot r, r^2 \cdot r\} = \{r, r^2, 1\}$$

$$Hr^2 = \{1 \cdot r^2, r \cdot r^2, r^2 \cdot r^2\} = \{r^2, 1, r\}$$

SAME

$$Hs = \{1 \cdot s, r \cdot s, r^2 \cdot s\}$$

$$= \{s, sr^{-1}, sr^{-2}\}$$

$$= \{s, sr^{3-1}, sr^{3-2}\}$$

$$= \{s, sr^2, sr\}$$

$$Hsr = \{1 \cdot sr, r \cdot sr, r^2 \cdot sr\}$$

SAM

$$= \{ sr, sr^{-1}r, sr^{-2}r \}$$

$$= \{ sr, s, \underbrace{sr^2}_{sr^{-1}} \}$$

$$Hsr^2 = \{ 1 \cdot sr^2, r \cdot sr^2, r^2 \cdot sr^2 \}$$

$$= \{ sr^2, sr^{-1}r^2, sr^{-2}r^2 \}$$

$$= \{ sr^2, sr, s \}$$

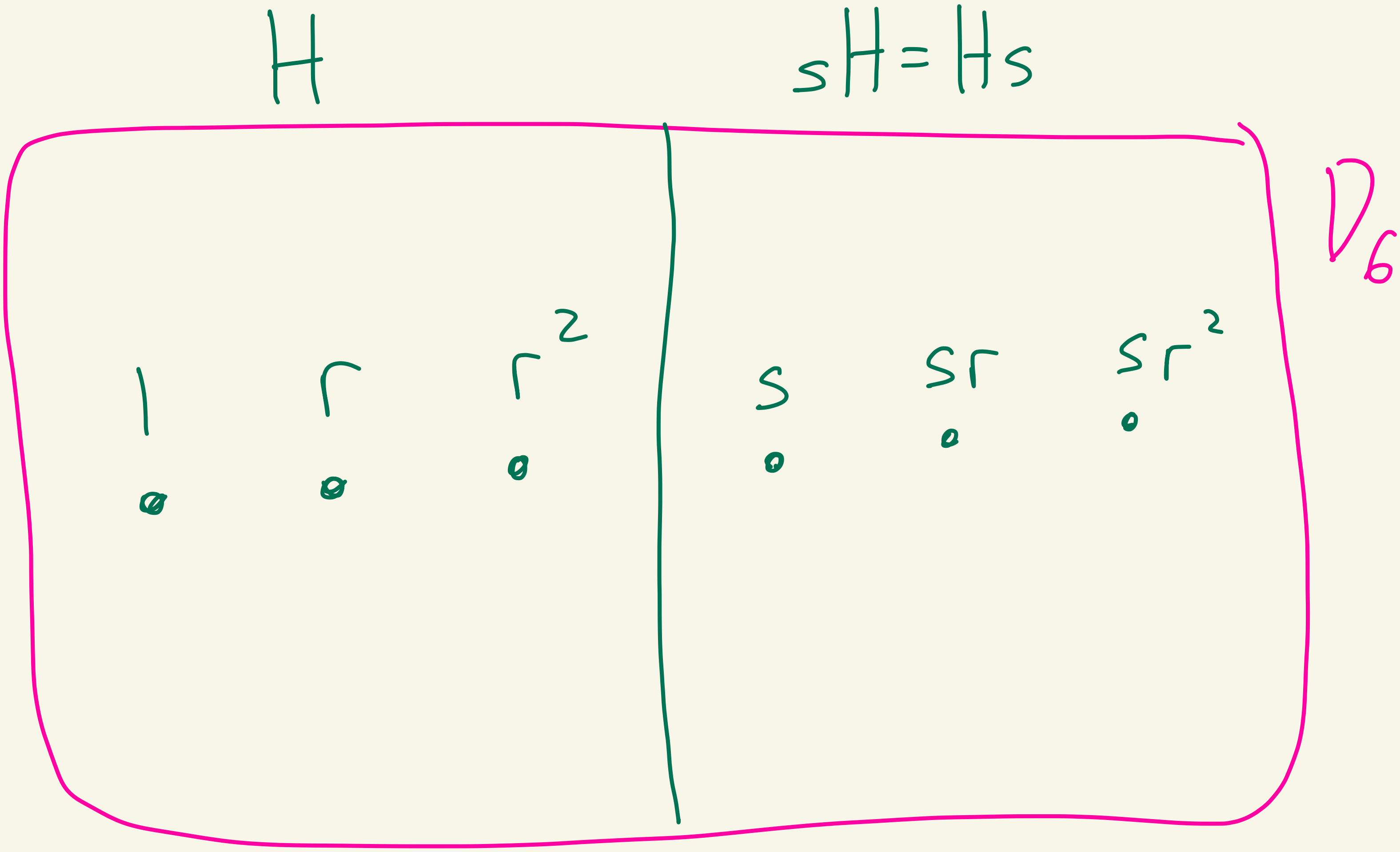
E

The right cosets are

$$\{1, r, r^2\} = H = Hr = Hr^2$$

$$\{s, sr, sr^2\} = Hs = Hsr = Hsr^2$$

<u>Summary:</u> The left and right cosets are the same. They partition the group D_6 into two pieces.



The set of left-cosets is

$$D_6/H = \{H, sH\}$$