Math 4550
10/15/25

Ex: Find <u>all</u> homomorphisms

$\varphi : U_6 \to U_4$

We have

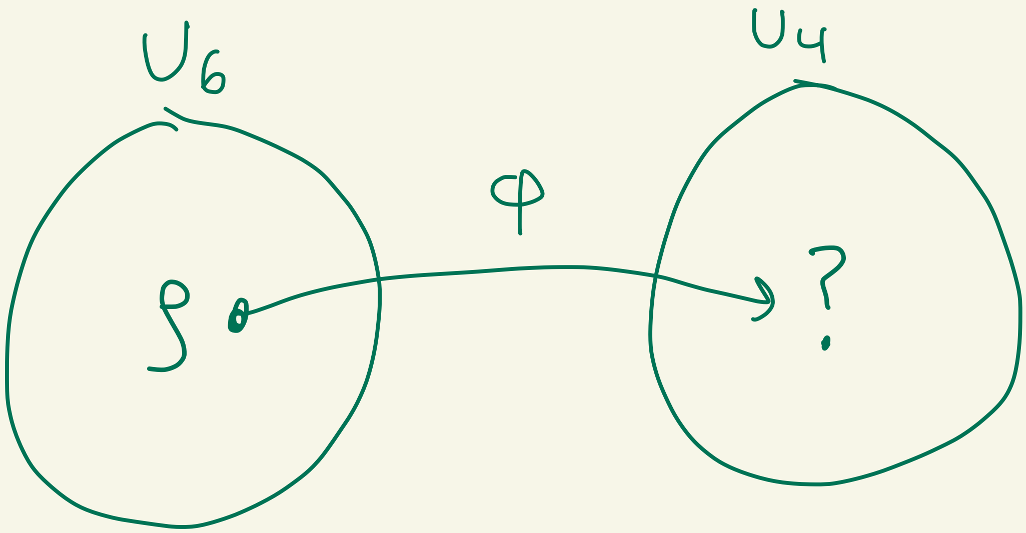
$U_6 = \{1, \rho, \rho^2, \rho^3, \rho^4, \rho^5\}$ where $\rho^6 = 1$

$\rho = e^{\frac{2\pi i}{6}}$

$U_4 = \{1, \gamma, \gamma^2, \gamma^3\}$ where $\gamma^4 = 1$

$\gamma = e^{2\pi i/4}$

We want all homomorphisms $\varphi : U_6 \to U_4$



$U_6 = \langle \rho \rangle$

First step is determine what $\varphi(\rho)$ is.

The order of ρ is 6.

So, $\varphi(\rho)$ has to have order dividing 6.

So, $\varphi(\rho)$ has to have order 1, 2, 3, or 6.

elements of U_4	order
1	1
γ	4
γ^2	2
γ^3	4

$\gamma \cdot \gamma^3 = \gamma^4 = 1$

$\gamma^{-1} = \gamma^3$

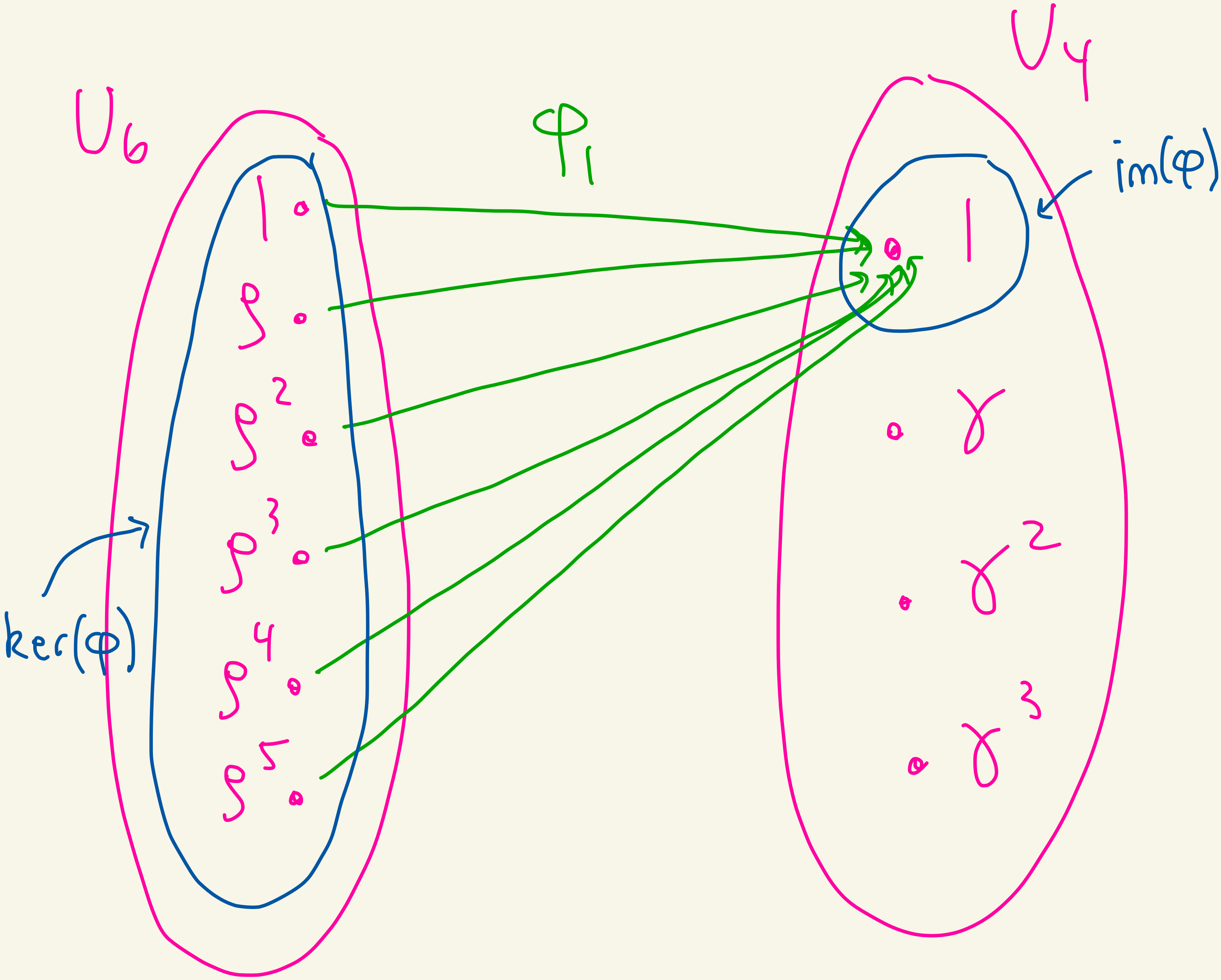
γ, γ^3 have same order

$\boxed{\gamma \text{ has order } 4}$

$\boxed{(\gamma^2)^2 = \gamma^4 = 1}$

$$\boxed{\text{We have two possibilities: } \varphi(\rho) = 1 \text{ or } \varphi(\rho) = \gamma^2}$$

<u>Case 1: $\varphi_1(\rho) = 1$</u>



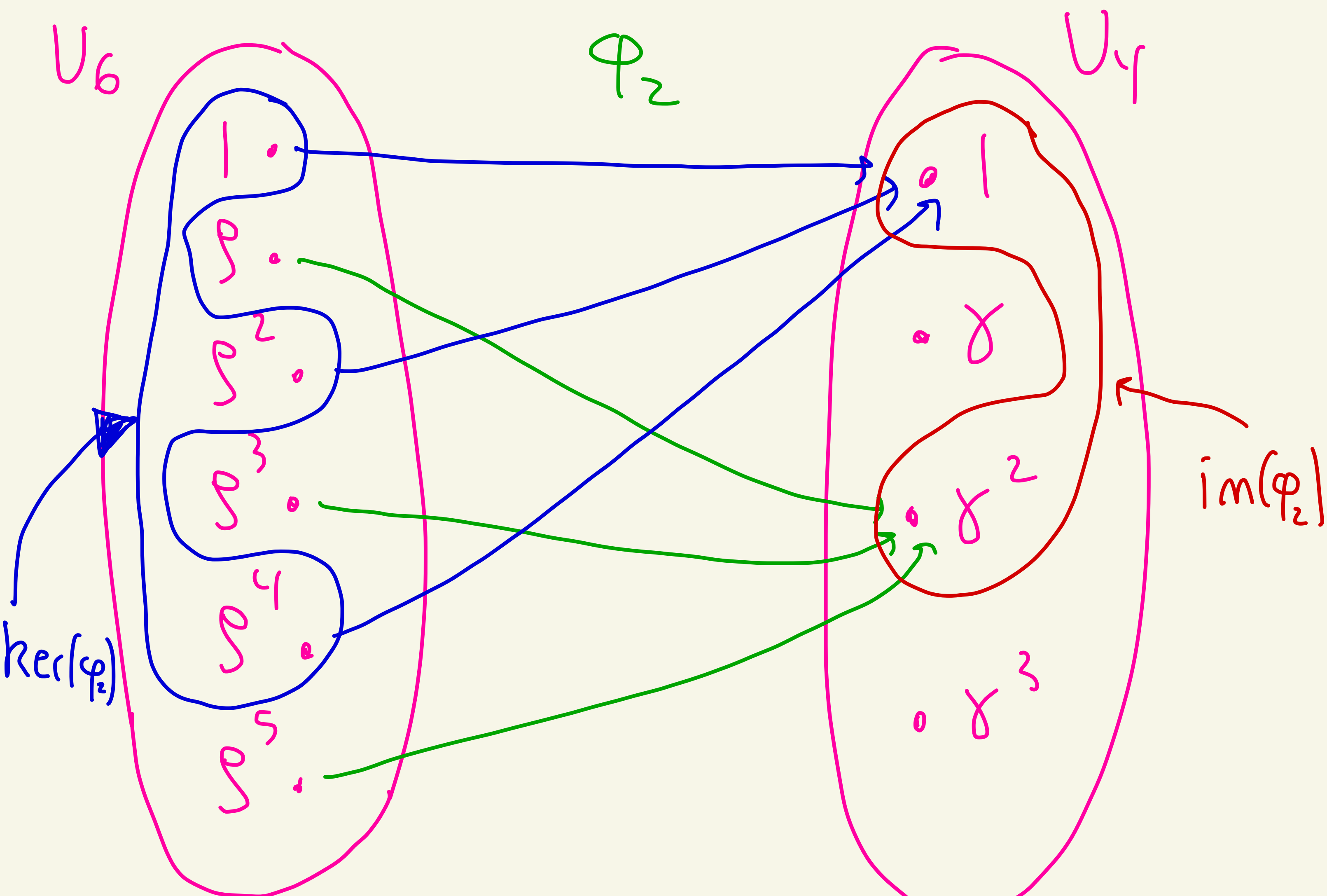
(called trivial homomorphism)

To make φ into a homomorphism need:

$$\varphi_1(\rho^2) = \varphi_1(\rho)\varphi_1(\rho) = 1 \cdot 1 = 1 = 1^2$$

$$\varphi_1(\rho^3) = \varphi_1(\rho)\varphi_1(\rho)\varphi_1(\rho) = 1 \cdot 1 \cdot 1 = 1 = 1^3$$

Case 2: $\varphi_2(\rho) = \gamma^2$



Calculations:

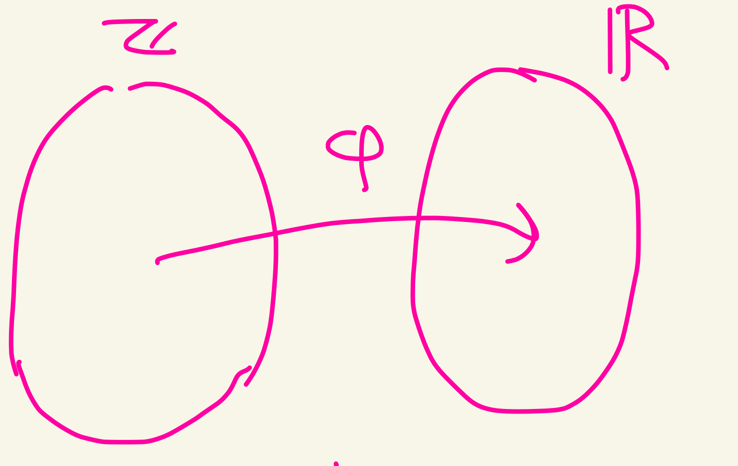
Use $\gamma^4 = 1$

$$\varphi_2(\rho^2) = \varphi_2(\rho)\varphi_2(\rho) = \gamma^2 \cdot \gamma^2 = \gamma^4 = 1$$

$$\varphi_2(\rho^3) = \varphi_2(\rho)\varphi_2(\rho)\varphi_2(\rho) = \gamma^2 \cdot \gamma^2 \cdot \gamma^2$$
$$= 1 \cdot \gamma^2 = \gamma^2$$

$\ker(\varphi_2) = \{1, \rho^2, \rho^4\}$ $\quad$ $\operatorname{im}(\varphi_2) = \{1, \gamma^2\}$

Ex: Let's construct a homomorphism $\varphi : \mathbb{Z} \to \mathbb{R}$.



$\mathbb{Z}$ is cyclic

$\mathbb{Z} = \langle 1 \rangle$

$\mathbb{Z} \leftarrow \boxed{+}$

$\mathbb{R} \leftarrow \boxed{+}$

$\mathbb{Z}$ is cyclic

$\mathbb{Z} = \langle 1 \rangle$

$\mathbb{Z} = \{\ldots, (-1)+(-1)+(-1), (-1)+(-1), (-1), 0, 1, 1+1, 1+1+1, \ldots\}$

Since $\mathbb{Z}$ is infinite we can send 1 anywhere we want!

Let's do $\boxed{\varphi(1) = e}$.

This will force where everything

else goes in order to make φ a homomorphism.

For example,

$$\varphi(2) = \varphi(1+1) = \varphi(1) + \varphi(1) = e+e = 2e$$

$$\varphi(3) = \varphi(1+1+1) = \varphi(1)+\varphi(1)+\varphi(1) = e+e+e = 3e$$

$$\varphi(4) = 4e$$

$$\varphi(5) = 5e$$

$$\vdots \qquad \vdots$$

What about $\varphi(0)$?

$$\varphi(0) = 0$$

identity goes to identity

What abut negative integers?

$$\varphi(-1) = -\varphi(1) = -e$$

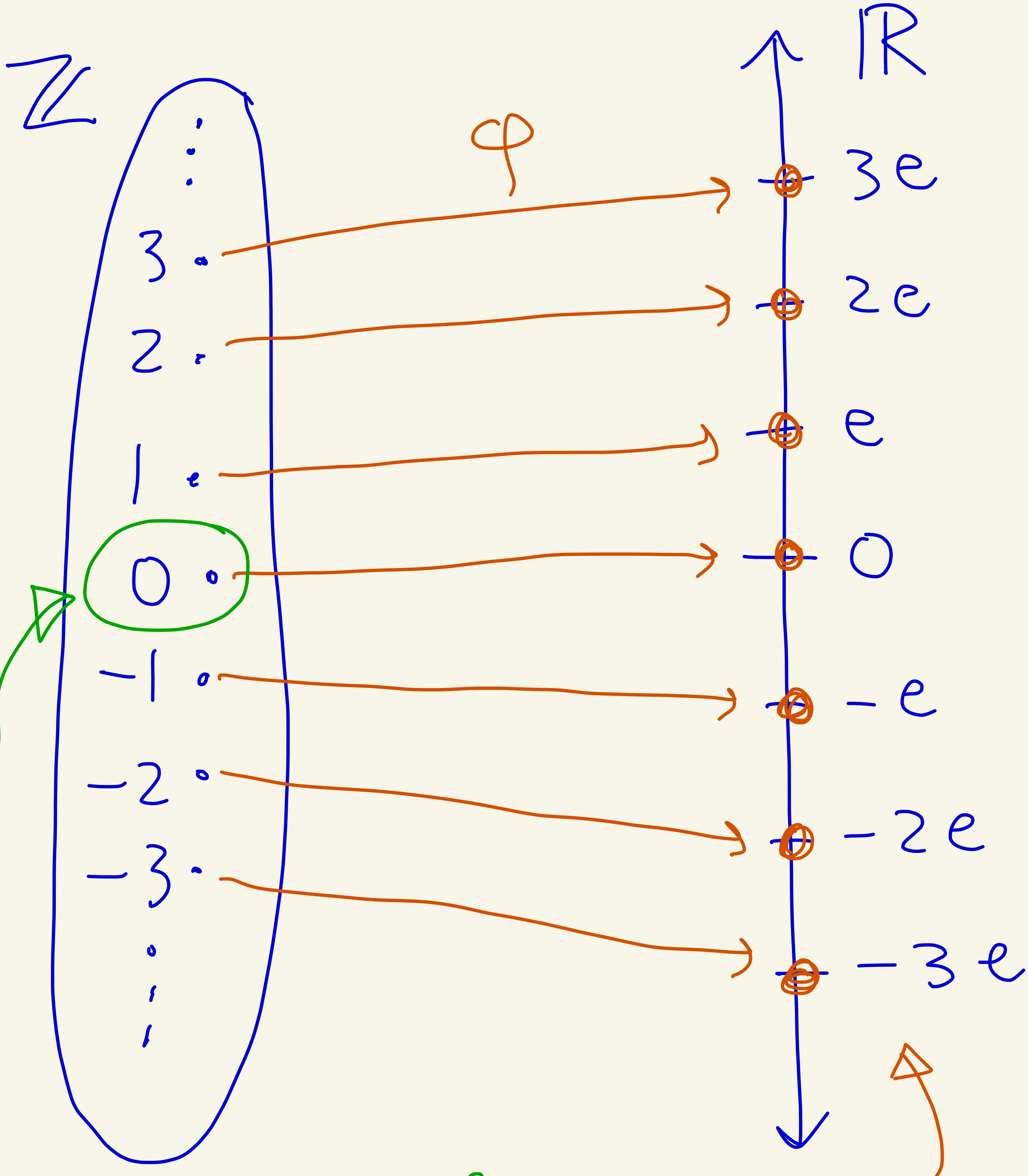
$$\boxed{\varphi(x^{-1}) = [\varphi(x)]^{-1}}$$

topic 4

$$\varphi(-2) = \varphi(-1) + \varphi(-1) = (-e) + (-e) = -2e$$

$$\varphi(-3) = \varphi(-1) + \varphi(-1) + \varphi(-1) = -e-e-e = -3e$$

In general, $\varphi(n) = ne$ for $n \in \mathbb{Z}$



$\ker(\varphi) = \{0\}$

$\operatorname{im}(\varphi) = \langle e \rangle = \{\ldots, -2e, -e, 0, e, 2e, \ldots\}$

HW Problem

$\mathbb{Q} = \left\{ \frac{a}{b} \;\middle|\; a, b \in \mathbb{Z}, b \neq 0 \right\}$

is not cyclic

$\boxed{\mathbb{Q} = \text{rational \#s}}$

proof:

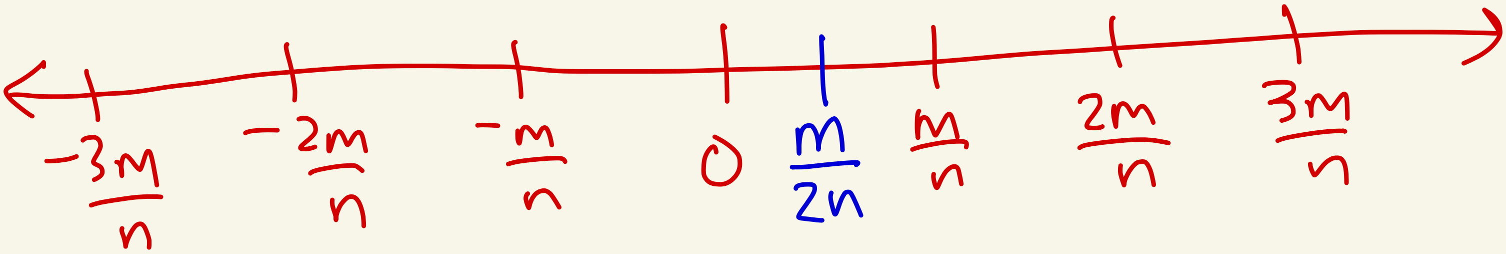
Suppose $\mathbb{Q}$ is cyclic.

Then $\mathbb{Q} = \left\langle \frac{m}{n} \right\rangle$ where $\frac{m}{n} \neq 0$.

So,

$\boxed{\left(-\frac{m}{n}\right) + \left(-\frac{m}{n}\right)}$ $\qquad$ $\boxed{\frac{m}{n} + \frac{m}{n}}$

$$\mathbb{Q} = \left\langle \frac{m}{n} \right\rangle = \left\{ \ldots, -\frac{3m}{n}, -\frac{2m}{n}, -\frac{m}{n}, 0, \frac{m}{n}, \frac{2m}{n}, \frac{3m}{n}, \ldots \right\}$$



This is nonsense, $\frac{m}{2n} \in \mathbb{Q}$ but

$$\frac{m}{2n} \notin \left\langle \frac{m}{n} \right\rangle,$$

Contradiction

$\mathbb{Q}$ is not cyclic! $\blacksquare$