Math 4550
10/1/25

Schedule

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Topic 5 - Cyclic groups

Theorem

Let G be a cyclic group.
If $H \leq G$, then H is cyclic

proof:

Since G is cyclic,

$$G = \langle x \rangle = \{ x^k \mid k \in \mathbb{Z} \}$$
$$= \{ \ldots, x^{-2}, x^{-1}, e, x, x^2, \ldots \}$$

where $x \in G$.

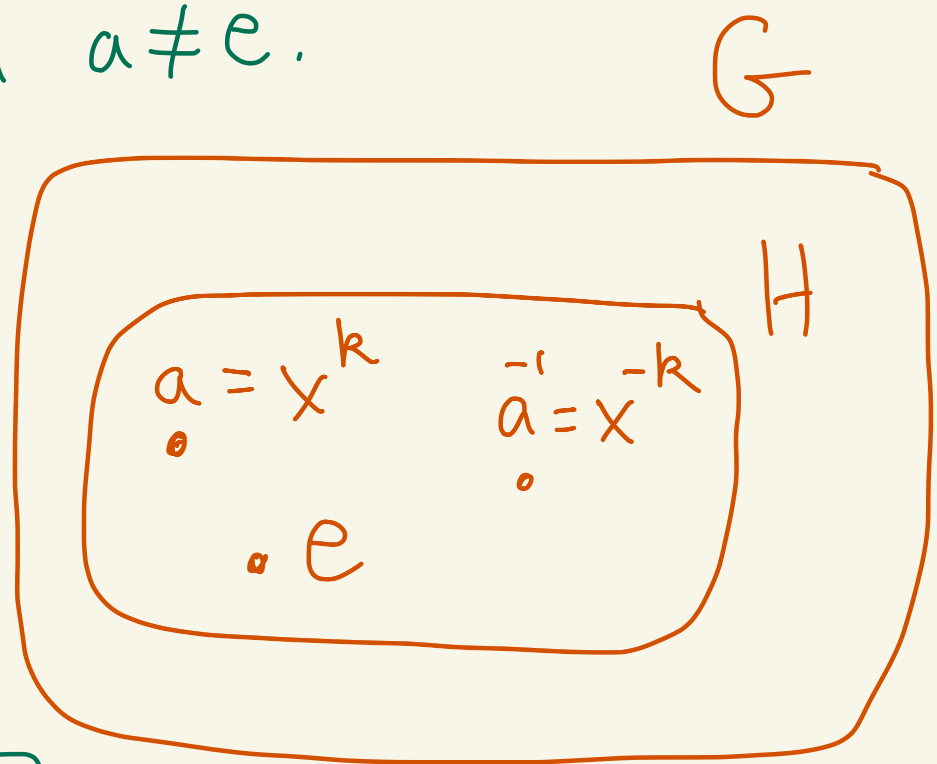
Let $H \leq G$.

We show that H is cyclic.

If $H = \{e\}$, then $H = \langle e \rangle$.

Now suppose $H \neq \{e\}$.

Then there exists $a \in H$ with $a \neq e$.



Since $G = \langle x \rangle$ we know $a = x^k$ where $k \neq 0$.

[$k \neq 0$ since $a \neq e$]

Note that $a^{-1} = (x^k)^{-1} = x^{-k} \in H$ because $a \in H$ and $H \leq G$.

Since $k \neq 0$, either $k > 0$ or $-k > 0$.

So H contains some x^n where n is a positive integer.

Let m be the smallest positive integer where $x^m \in H$.

$\boxed{\underline{\text{Claim:}}\ H = \langle x^m \rangle}$

<u>pf of claim:</u>

Note that $\langle x^m \rangle \subseteq H$,
because $x^m \in H$, and
thus $(x^m)^l \in H$ for any $l \in \mathbb{Z}$
because $H \leq G$.

Now let's show $H \subseteq \langle x^m \rangle$.

Let $y \in H$.

Since $H \leq G$ and $G = \langle x \rangle$
we know $y = x^f$ where $f \in \mathbb{Z}$.

Divide m into f to get
$$f = qm + r$$

where $q, r \in \mathbb{Z}$ and $0 \leq r < m$.

So, $x^f = x^{qm+r} = x^{qm} x^r$

Thus, $x^r = x^{-qm} x^f$

$$= (\underbrace{x^m}_{x^m \in H})^{-q} \underbrace{x^f}_{y = x^f \in H} \in H$$

reasoning: $x^m \in H \to (x^m)^{-q} \in H$

$(x^m)^{-q} \in H,\ x^f \in H \longrightarrow (x^m)^{-q} x^f \in H$

All because $H \leq G$.

So, $x^r \in H$ and $0 \leq r < m$.

But m is the smallest positive power of x that is in H.

Thus, $r = 0$ ← r can't be positive

Hence,

$$f = qm + r = qm + 0 = qm.$$

Thus,

$$y = x^f = x^{qm} = (x^m)^q.$$

Ergo,

$$H \subseteq \langle x^m \rangle.$$

Therefore,

$$H = \langle x^m \rangle. \quad \boxed{}$$

Ex: Find all subgroups of

$\mathbb{Z}_{12} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}, \bar{8}, \bar{9}, \overline{10}, \overline{11}\}$

Since $\mathbb{Z}_{12} = \langle \bar{1} \rangle$ is cyclic we know all the subgroups are cyclic.

HW: G is a group, $x \in G$.
Then: $\langle x \rangle = \langle x^{-1} \rangle$

All subgroups of $\mathbb{Z}_{12}$:

$\langle \bar{0} \rangle = \{\bar{0}\}$

$\langle \bar{1} \rangle = \{\bar{0}, \bar{1}, \bar{2}, \ldots, \overline{11}\} = \mathbb{Z}_{12}$

$\langle \bar{2} \rangle = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \overline{10}\}$

$$\langle \overline{3} \rangle = \{ \overline{0}, \overline{3}, \overline{6}, \overline{9} \}$$

$$\langle \overline{4} \rangle = \{ \overline{0}, \overline{4}, \overline{8} \}$$

$$\langle \overline{5} \rangle = \{ \overline{0}, \overline{5}, \overline{10}, \underset{\boxed{\overline{15}}}{\overline{3}}, \overline{8}, \underset{\boxed{\overline{13}}}{\overline{1}}, \overline{6}, \overline{11}, \underset{\boxed{\overline{16}}}{\overline{4}}, \overline{9}, \underset{\boxed{\overline{14}}}{\overline{2}}, \overline{7} \} = \mathbb{Z}_{12}$$

$$\langle \overline{6} \rangle = \{ \overline{0}, \overline{6} \}$$

$$\langle \overline{7} \rangle = \langle \overline{5} \rangle = \mathbb{Z}_{12}$$

$$\boxed{\overline{7}^{-1} = \overline{5} \text{ since } \overline{7} + \overline{5} = \overline{0}}$$

$$\langle \overline{8} \rangle = \langle \overline{4} \rangle = \{ \overline{0}, \overline{4}, \overline{8} \}$$

$$\boxed{\overline{8}^{-1} = \overline{4}}$$

$$\boxed{\overline{10}^{-1} = \overline{2}}$$

$$\langle \overline{9} \rangle = \langle \overline{3} \rangle, \quad \langle \overline{10} \rangle = \langle \overline{2} \rangle,$$

$$\boxed{\overline{9}^{-1} = \overline{3}}$$

$$\langle \overline{11} \rangle = \langle \overline{1} \rangle.$$

$\boxed{\overline{11}^{-1} = \overline{1}}$

$\boxed{\text{Subgroup diagram}}$

$$
\begin{tikzcd}[column sep=small]
& \mathbb{Z}_{12} \arrow[dl, dash] \arrow[dr, dash] & \\
\{\overline{0},\overline{2},\overline{4},\overline{6},\overline{8},\overline{10}\} \arrow[d, dash] & & \{\overline{0},\overline{3},\overline{6},\overline{9}\} \arrow[d, dash] \\
\{\overline{0},\overline{4},\overline{8}\} \arrow[dr, dash] & & \{\overline{0},\overline{6}\} \arrow[dl, dash] \\
& \{\overline{0}\} &
\end{tikzcd}
$$

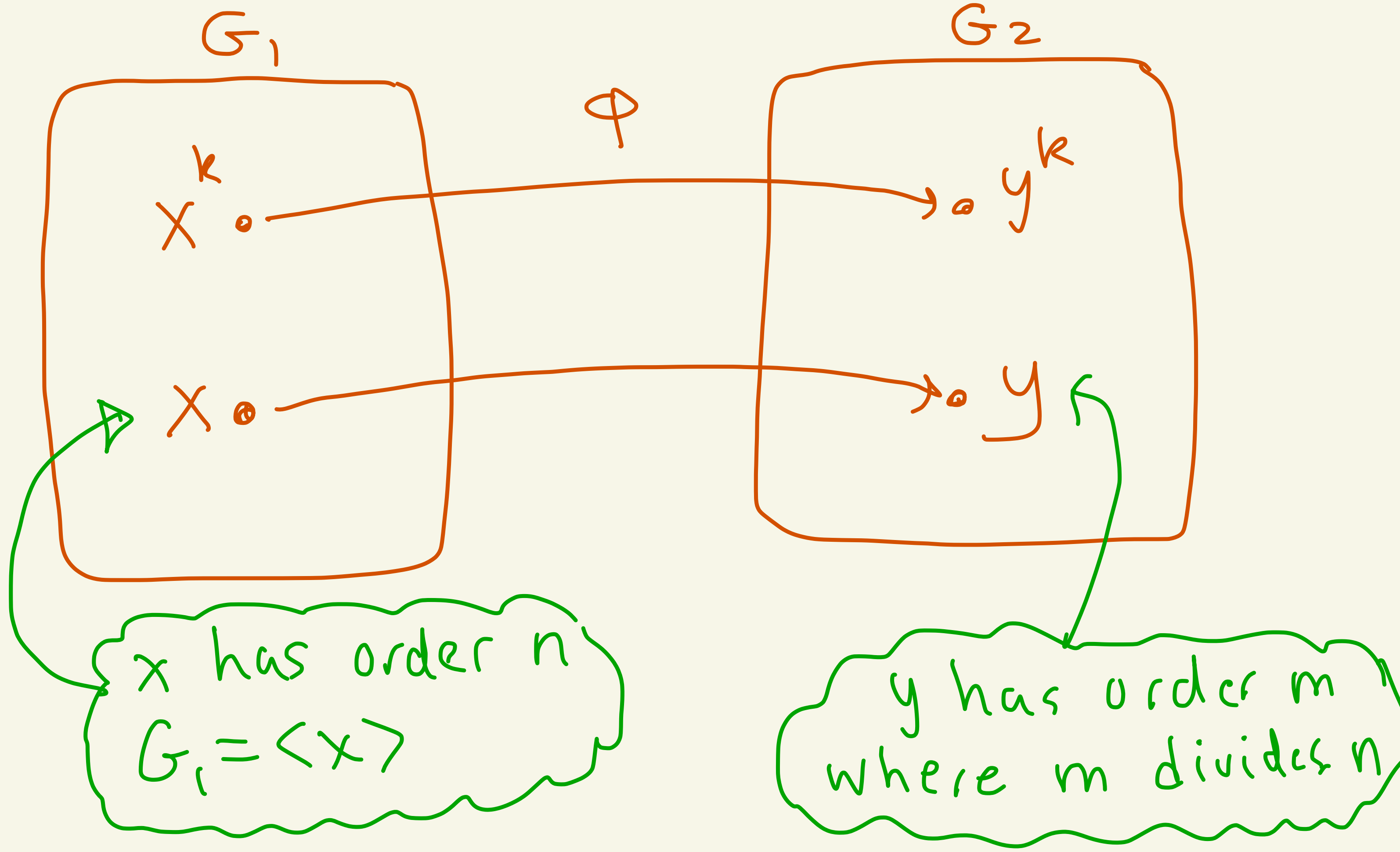
$$
\boxed{\begin{array}{c} A \\ | \\ B \\ \text{means} \\ B \leq A \end{array}}
$$

<u>Theorem</u> (Homomorphisms out of a cyclic group)

Let $G_1 = \langle x \rangle$ be a cyclic group. Let G_2 be any group. Let's classify all homomorphisms $\varphi: G_1 \to G_2$.

$\boxed{\text{case 1: Suppose } x \text{ has finite order } n}$

Pick $y \in G_2$ with order m dividing n. Then, $\varphi: G_1 \to G_2$ given by $\varphi(x^k) = y^k$ is a homomorphism. Furthermore, every homomorphism from G_1 to G_2 is of this form.



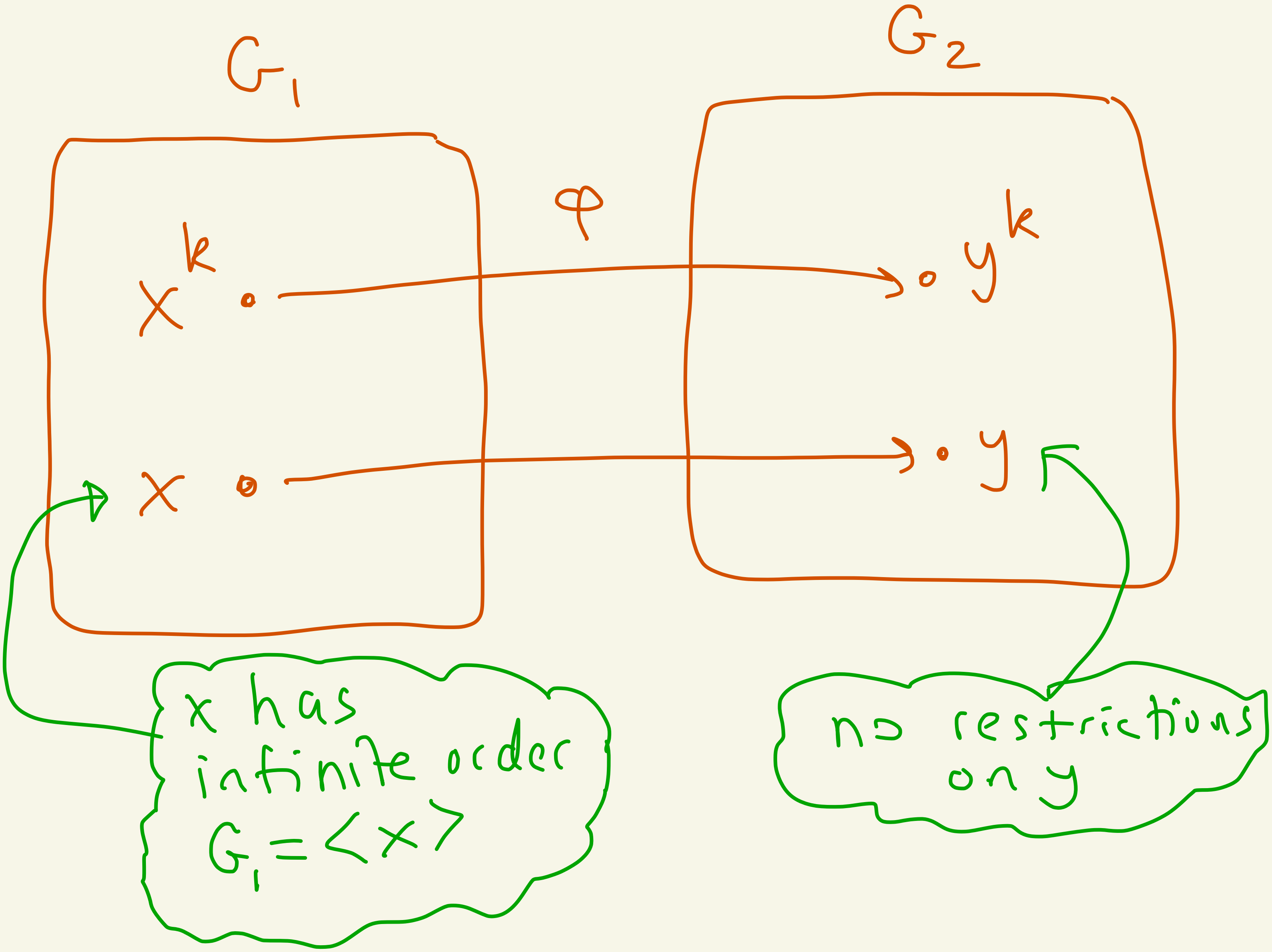
$\boxed{\text{Case 2: Suppose } x \text{ has infinite order}}$

Let $y \in G_2$ ← no restrictions on y

Define $\varphi : G_1 \to G_2$ by $\varphi(x^k) = y^k$.

Then φ is a homomorphism.

Furthermore, any homomorphism from G_1 to G_2 is of this form.



G_1
G_2
φ
x^k
x
y^k
y
x has infinite order
$G_1 = \langle x \rangle$
no restrictions on y