Math 4460

4/19/23

M	W
	4/19 topic 6
4/24 review	4/26 test 2
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$\boxed{\text{Topic 6 – Gaussian integers}}$

Recall that the set of complex numbers is

$$\mathbb{C} = \{ x + iy \mid x, y \in \mathbb{R} \}$$

$$= \{ 5, 1+i, \tfrac{1}{2}i, \pi + e^2 i, \dots \}$$

$5 + 0i$

where $\boxed{i^2 = -1}$ $\leftarrow$ $\boxed{\text{"} i = \sqrt{-1} \text{"}}$

Here are examples of adding and multiplying:

$$(1-2i)+\left(\frac{1}{2}+4i\right)$$

$$=\frac{3}{2}+2i$$

and

$$(1-2i)\left(\frac{1}{2}+4i\right)$$

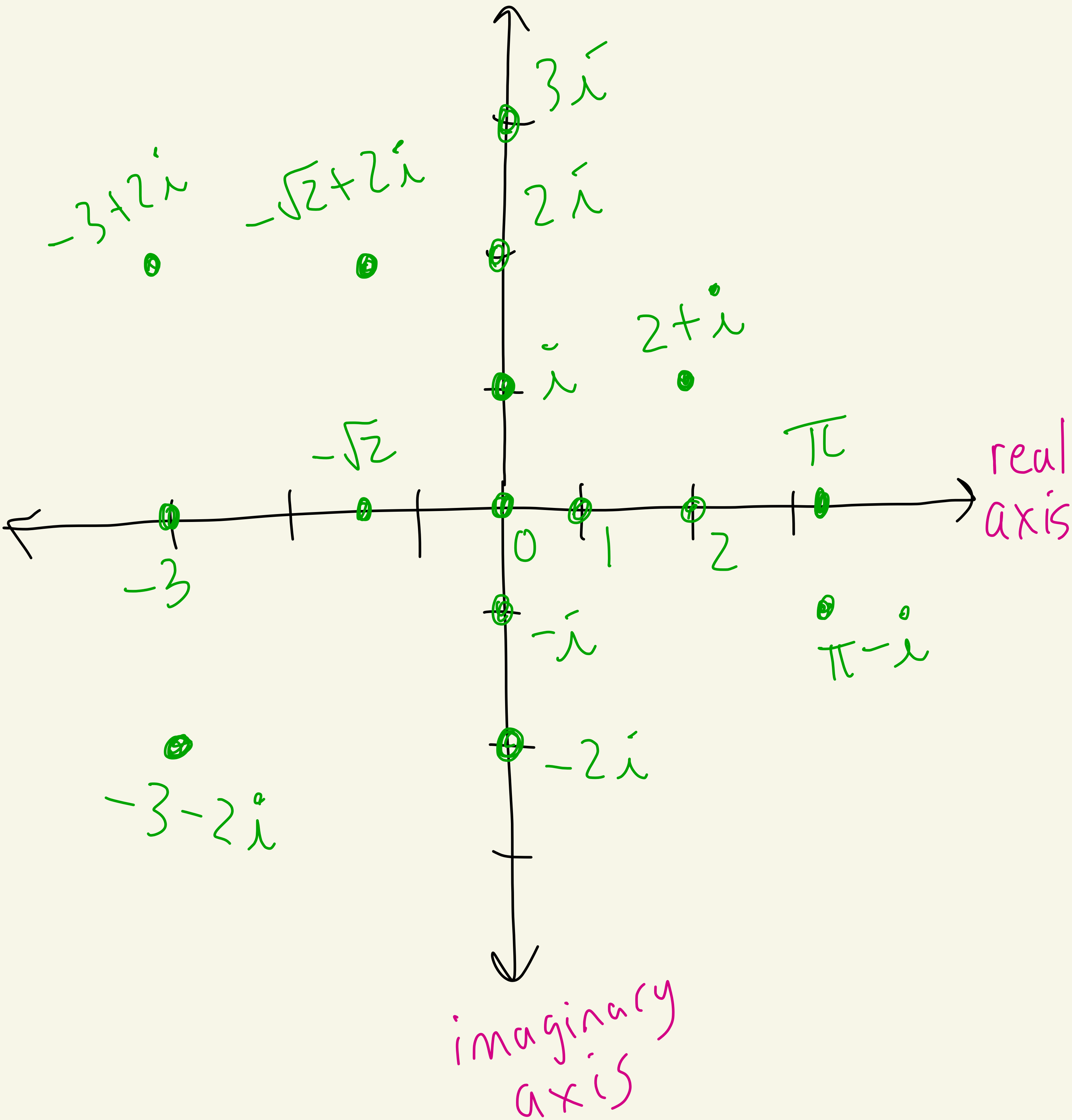
$$=\frac{1}{2}+4i-i-8i^2$$

$$=\frac{1}{2}+3i-8(-1)$$

$$=\frac{17}{2}+3i$$

Picture of $\mathbb{C}$

$x+iy$ located at the (x,y) spot



Def: Let $z = x + iy$ be a complex number.

The <u>conjugate</u> of z is $\bar{z} = x - iy$.

$$\bar{z} = x - iy$$

The <u>absolute value</u> of z is $|z| = \sqrt{x^2+y^2}$

The <u>real part</u> of z is x and we write $Re(z) = x$

The <u>imaginary part</u> of z is y and we write $Im(z) = y$

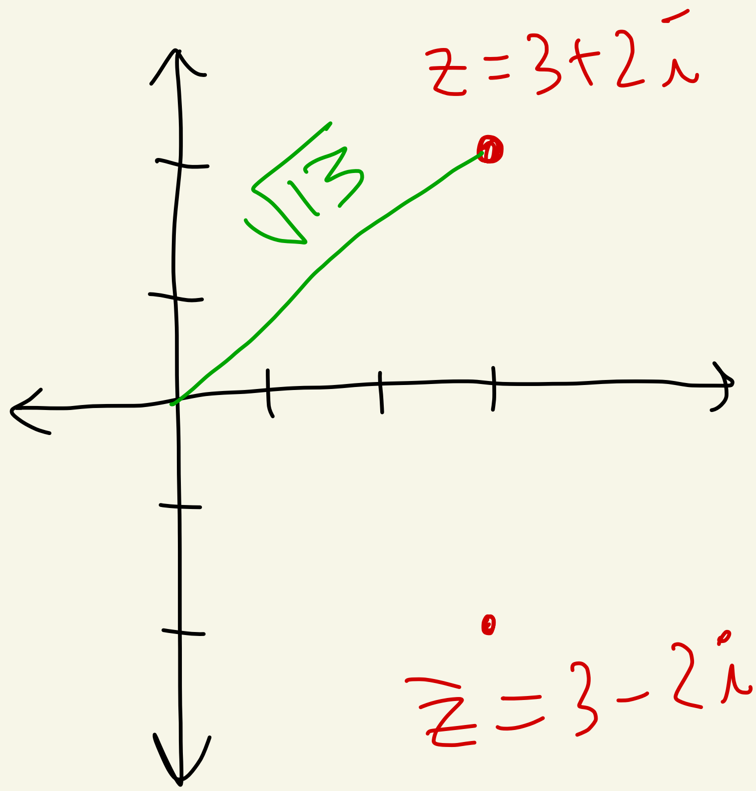
Ex: $z = 3 + 2i$

$\bar{z} = 3 - 2i$

$|z| = \sqrt{3^2 + (-2)^2}$
$= \sqrt{13}$

$Re(3+2i) = 3$

$Im(3+2i) = 2$



Division in $\mathbb{C}$

$$\frac{a+bi}{c+di} = \left(\frac{a+bi}{c+di}\right)\left(\frac{c-di}{c-di}\right)$$

$$= \cdots\cdot$$

Ex:

$$\frac{2+i}{1-2i} = \left(\frac{2+i}{1-2i}\right)\left(\frac{1+2i}{1+2i}\right)$$

$$= \frac{2+4i+i+2i^2}{1+\cancel{2i}-\cancel{2i}-4i^2}$$

$i^2=-1$

$$= \frac{2+5i-2}{1+4}$$

$$= \frac{5i}{5} = i$$

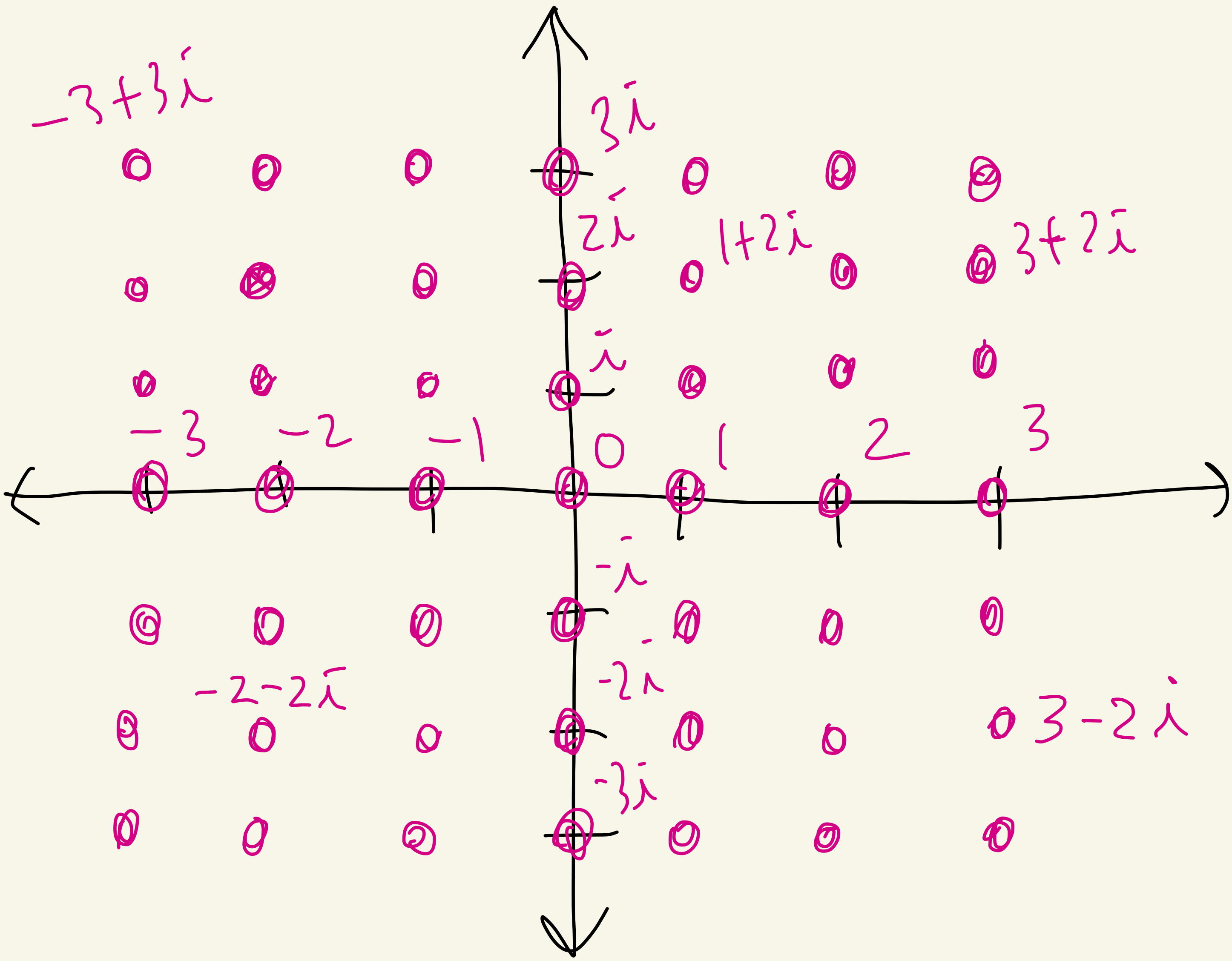
So,

$$\frac{2+i}{1-2i} = i$$

<u>Def:</u> The set

$$\mathbb{Z}[i] = \{a+bi \mid a,b \in \mathbb{Z}\}$$

is called the <u>Gaussian integers</u>.



<u>Note:</u> $\mathbb{Z} \subseteq \mathbb{Z}[i]$

<u>Note:</u> If $z, w \in \mathbb{Z}[i]$, then $z+w \in \mathbb{Z}[i]$

and $z \cdot w \in \mathbb{Z}[i]$.

That is, $\mathbb{Z}[i]$ is closed under addition and multiplication.

<u>proof:</u> Let $z = a+bi$, $w = c+di$ where $a, b, c, d \in \mathbb{Z}$.

Then,

$$z + w = \underbrace{(a+c)}_{\text{in } \mathbb{Z}} + \underbrace{(b+d)}_{\text{in } \mathbb{Z}} i$$

$$\begin{aligned} z \cdot w &= ac + bdi^2 + adi + bci \\ &= \underbrace{(ac-bd)}_{\text{in } \mathbb{Z}} + \underbrace{(ad+bc)}_{\text{in } \mathbb{Z}.} i. \end{aligned}$$

$\blacksquare$

<u>Note:</u> $\mathbb{Z}[i]$ is not closed under division. For example,

$$\frac{1-2i}{2+3i} = \frac{1-2i}{2+3i}\cdot\frac{2-3i}{2-3i}$$

$$= \frac{2-3i-4i+6i^2}{4-\cancel{6i}+\cancel{6i}-9i^2}$$

$i^2=-1$

$$= \frac{2-7i-6}{4+9}$$

$$= \frac{-4}{13} - \frac{7}{13}i \notin \mathbb{Z}[i]$$

<u>Def:</u> Let $z = x+iy \in \mathbb{Z}[i]$

The <u>norm</u> of z is

$$N(z) = x^2+y^2$$

Note: $z\bar{z} = (x+iy)(x-iy)$
$= x^2 - \cancel{ixy} + \cancel{ixy} - i^2 y^2$
$= x^2 + y^2 = N(z)$

Ex: $N(2+3i) = 2^2 + 3^2 = 13$

$N(5-2i) = 5^2 + (-2)^2 = 29$

$N(2) = N(2+0i) = 2^2 + 0^2 = 4$

$N(-3i) = N(0-3i) = 0^2 + (-3)^2 = 9$

Theorem: Let $z, w \in \mathbb{Z}[i]$.

Then:

① $N(z)$ is an integer and $N(z) \geq 0$

② $N(z) = 0$ iff $z = 0$

③ $N(zw) = N(z)N(w)$

proof: Let $z = a + bi$ and $w = c + di$ where $a, b, c, d \in \mathbb{Z}$.

① $N(z) = a^2 + b^2$ is a non-negative integer

② $N(z) = a^2 + b^2 = 0$ iff $a = b = 0$

iff $z = a + ib = 0$

3 $N(zw)$

$= N((a+bi)(c+di))$

$= N(ac + adi + bci + bdi^2)$

$= N((ac-bd)+(ad+bc)i)$

$= (ac-bd)^2 + (ad+bc)^2$

$= a^2c^2 - \cancel{2abcd} + b^2d^2$
$+ a^2d^2 + \cancel{2abcd} + b^2c^2$

$= a^2(c^2+d^2) + b^2(c^2+d^2)$

$= (a^2+b^2)(c^2+d^2)$

$= N(z)\cdot N(w)$ $\boxed{/\!/\!/}$

__Conceptual:__ Think of the norm function as a way to map $\mathbb{Z}[i]$ to the non-negative integers

$$\boxed{N(a+bi) = a^2+b^2}$$