Math 4460 3/22/23

Summary: There are three kinds of Pythagorean triples. $(1)(\chi, 0, \pm \chi)$ E_{χ} : (x,y,2) = (3,0,3) $\chi^2 + O^2 = (\pm \chi)^2$ 2(0, y, ty) $\frac{Ex:}{(x,y,z) = (0,5,-5)}$ $0^{2} + y^{2} = (\pm y)^{2}$ (3) the ones that are multiples of positive, primitive Pythagorean triples with possible sign adjustments (5,12,13) XZ (10,24,26) Sign (-10,24,26)Primitive Primitive

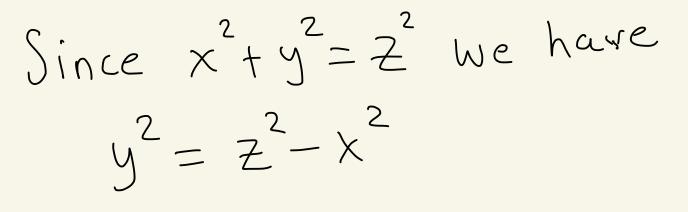
Let (X,Y,Z) be a positive, Primitive, Pythagorean triple. x > 0, y > 0, Z > 0, So, gcd(X,Y,Z)=1 $X + Y^2 = Z^2$. X,Y,ZEZ · Let's show x and y can't both be even. Why? Suppose x and y are both even. Then in $\mathbb{Z}_2 = \{\overline{0}, \overline{1}\}$ we would have $\overline{X} = \overline{O}$ and $\overline{y} = \overline{O}$. Then, $\overline{Z}^2 = \overline{Z}^2 = \chi^2 + \gamma^2$

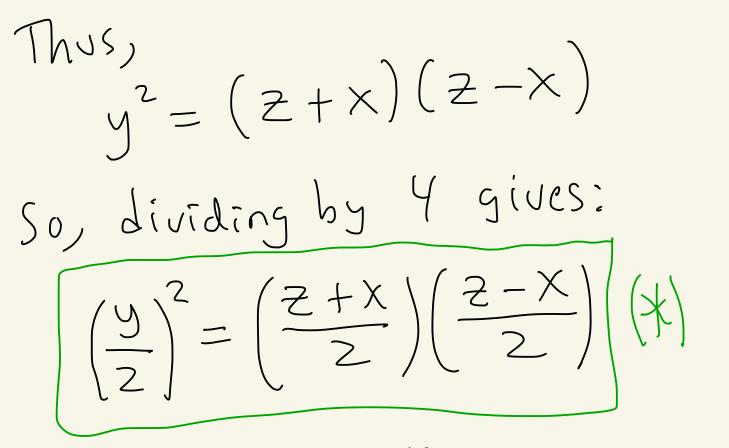
$$= \overline{\chi^{2}} + \overline{y^{2}}$$
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$$= \overline{\chi^{2}} + \overline{\chi^{2}} = \overline{\chi^{2}}.$$

Why? Suppose x and y are both odd Recall in $\mathbb{Z}_{Y} = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}\}$ a is odd $\overline{iff} \ \overline{a} = \overline{1} \ or \ \overline{a} = 3$ And in $\mathbb{Z}_{4}, T=T$ $and \overline{3}^2 = \overline{9} = \overline{1}.$ Thus, if x and y are both odd then $\overline{x}^2 = \overline{1}$ and $\overline{y}^2 = \overline{1}$. So, z = x + y = 1 + 1 = 2But this can't happen in Zy by this 222 table Ō 2 4=0

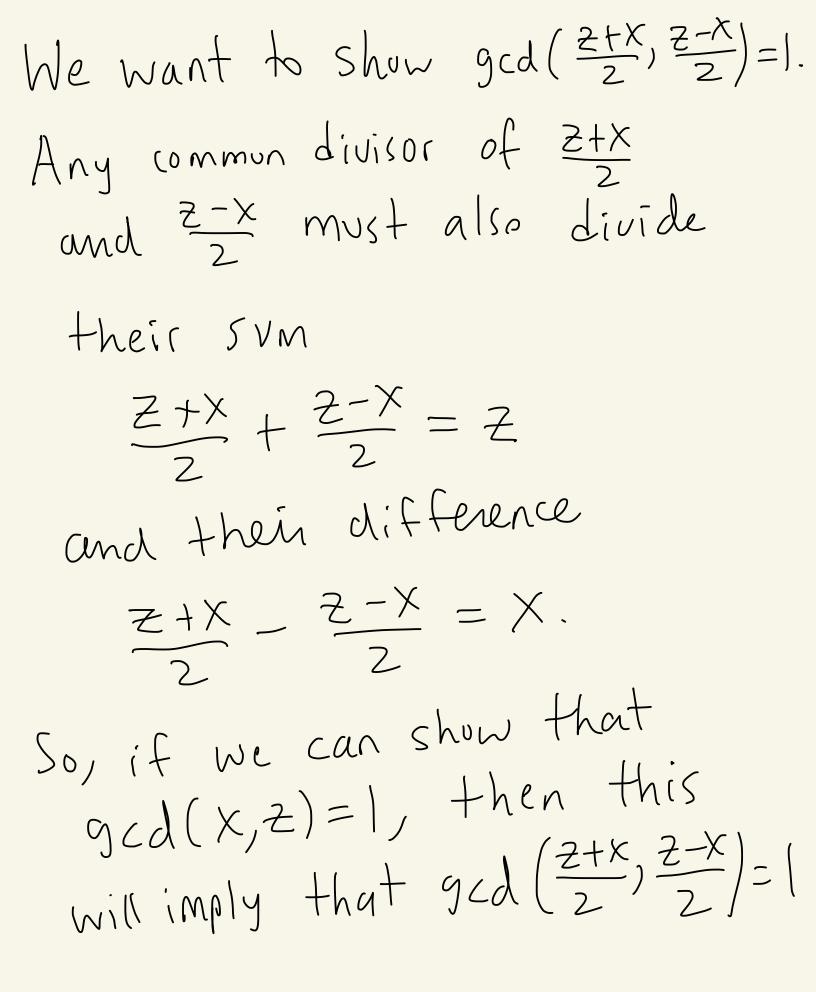
Let us assume that X is odd and y is even Then, in Z2= Zoji we have Z = X + Y $= 1 + 0^{2}$ = $S_0, \overline{Z}^2 = \overline{I} \quad in \quad \mathbb{Z}_2.$ Thus, Z=I in Z2. Hence Z is odd.

Since x is odd and Z is odd we know Z-x is even and Z+x is even.





Note: $\frac{y}{z}, \frac{z+x}{z}, \frac{z-x}{z} \in \mathbb{Z}$ since y, z+x, z-x are all even.



Why is gcd(x,z) = [?]HW 3-5(a) gcd(x,z) ≠1 iff there where exists a prime p plx and plz Suppose gcd(x,z) = (, Then by ItW, there exists a prime p where p | X and p/Z. Then, plx² and plz². So, $p|(z^2-x^2)$. $y^2=z^2-x^2$ Then $p|y^2$

Since p is prime
and ply.y
we know ply.
But then plx, ply, plz and

$$gcd(x,y,z) \ge p$$
.
Contradiction since $gcd(x,y,z)=1$.
Thus, $gcd(x,z)=1$
and so $gcd(\frac{z+x}{z},\frac{z-x}{z})=1$.
Recall this theorem: If A, B, C
are positive integers and
 $gcd(A,B)=1$ and $C^{*}=AB$
then there exist positive

integers R, S where topic $A = R^n$ and $B = S^n$ Section Our situation from (*) that ís thm $\left(\frac{y}{z}\right)^{L} = \left(\frac{z+x}{z}\right)\left(\frac{z-x}{z}\right)$ $C^2 = AB$ with $Gcd\left(\frac{2+x}{2}, \frac{2-x}{2}\right) = 1$ Hence, $\frac{z+x}{z} = r^2$ and $\frac{z-x}{z} = s^2$ where r, s are positive integers and gcd(r,s)=1 because $gcd\left(\frac{2+x}{2},\frac{2-x}{2}\right) = l$ So, $\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} z+x \\ z \end{pmatrix} \begin{pmatrix} z-x \\ z \end{pmatrix} = r^2 s^2$

y,r,s are positive. $S_{v}, \frac{y}{2} = \Gamma S.$ 0r, y=2rs $\frac{2}{2} = \frac{2+\chi}{2} > \frac{2-\chi}{2} = s^2.$ Note that $So, \Gamma75$ and Also, since Z is odd $Z = \frac{Z+X}{2} + \frac{Z-X}{2} = \Gamma + S^{2}$ We must have that \mathbb{Z}_2 r and s have opposite parity. That is, either r is odd & siseven

or ris even & sis odd.

Theorem: If (x,y,z) is a positive, primitive Pythagorean triple, with y even, then $\chi = r^2 - s^2 + \left(x = \frac{2+x}{2} - \frac{2-x}{2}\right)$ y = 2rs $Z = r^2 + S^2$ where r and s me of positive integers r>s70 opposite painty and and ycd(r,s) = l

S	r	$X = r^2 - S^2$	y = 2rs	$Z=r+S^2$
	2	3	4	5
	4	15	8	17
	6	35	γ2	37
2	3	5) 2	13
2	5	21	20	29
3	4	7	24	25
0 Q 0	0 1 .		0	0 0 0