Math 4460
1/30/23

How are the primes spaced out?

2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13,
14, 15, 16, 17, 18, 19, 20, 21, 22,
23, 24, 25, 26, 27, 28, 29, 30, 31,
32, 33, 34, 35, 36, 37, 38, 39,
40, 41, 42, 43, 44, 45, 46, 47,
48, 49, 50, 51, 52, 53, 54, 55, 56,
57, 58, 59, 60, 61, 62, 63, 64, 65,
66, 67, 68, 69, 70, 71, 72, 73, 74,
75, 76, 77, 78, 79, 80, 81, 82,
83, 84, 85, 86, 87, 88, 89, 90,
91, 92, 93, 94, 95, 96, 97,
98, 99, 100, 101, ...

Let $N=4$

Find 4 consecutive composite integers.

$$(N+1)!+2,\ (N+1)!+3,\ (N+1)!+4,\ (N+1)!+\underbrace{5}_{\boxed{N+1}}$$

$$5!+2,\ 5!+3,\ 5!+4,\ 5!+5$$

$$\underbrace{5\cdot4\cdot3\cdot2+2}_{\text{2 divides this}},\ \underbrace{5\cdot4\cdot3\cdot2+3}_{\text{3 divides this}},\ \underbrace{5\cdot4\cdot3\cdot2+4}_{\text{4 divides this}},\ \underbrace{5\cdot4\cdot3\cdot2+5}_{\text{5 divides this}}$$

$$122,\ 123,\ 124,\ 125$$

We just made $N=4$ consecutive $\underbrace{\text{composite}}_{\text{not prime}}$ numbers.

Theorem: There are arbitrarily large gaps in the primes. That is, given any positive integer N there exist N consecutive composite integers.

Proof: Let N be a positive integer.

Consider the following consecutive integers:

$(N+1)!+2,\ (N+1)!+3,\ (N+1)!+4,\ \ldots,\ (N+1)!+(N+1)$

Pick $k \in \mathbb{Z}$ with $2 \le k \le N+1$.

Then,

$$(N+1)!+k =$$

$$= \underbrace{(N+1)\cdot N\cdot (N-1)\cdots(k+1)k(k-1)\cdots(3)(2)}_{(N+1)!} + k$$

$$= k\left[(N+1)N(N-1)\cdots(k+1)(k-1)\cdots(3)(2)+1\right]$$

So, $k \mid [(N+1)! + k]$.

Note $k \neq 1$ because $2 \leq k$.

And also,

$$k \leq N+1 \leq (N+1)! \underset{k \geq 2}{<} (N+1)! + k$$

So, $k \neq (N+1)! + k$.

Thus, k is a divisor of $(N+1)! + k$ where $k \neq 1$ and $k \neq (N+1)! + k$.

So, $(N+1)! + k$ is not prime.

Thus we have created a list of N consecutive composite integers.

$\blacksquare$

Ex: If you plug $N=8$ into the formulas you will get

k	$(N+1)!+k$
2	362,882
3	362,883
4	362,884
5	362,885
6	362,886
7	362,887
8	362,888
9	362,889

Here are 13 composites in a row that are WAY earlier:

114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126

Hw 1

(11) Let $n > 1$ be an integer.
If $2^n - 1$ is prime, then
n is prime.

prime

<u>Ex:</u> $2^{3} - 1 = 7$ is prime

<u>Ex:</u> $2^{11} - 1 = 2047 = 23 \cdot 89$
is not prime even though 11 is prime.
So the converse "If n is prime, then
$2^n - 1$ is prime" is false.

<u>proof:</u> Suppose $n > 1$
Let's prove the contrapositive:
"If n is not prime, then $2^n - 1$

is not prime!"

Suppose n is not prime.
By HW 1 #7 we know

$$n = ab$$

where $1 < a$ and $1 < b$.

Then,

$$2^n - 1 = 2^{ab} - 1$$

$$= (2^a - 1)(2^{a(b-1)} + 2^{a(b-2)} + \cdots + 2^{2a} + 2^a + 1)$$

Ex:

$$2^{2 \cdot 4} - 1 = (2^2 - 1)(2^6 + 2^4 + 2^2 + 1)$$

$a = 2, b = 4$

Since $a \geq 2$ we know the first factor satisfies

$$2^a - 1 \geq 2^2 - 1 = 3.$$

Since $b \geq 2$ we know the second factor satisfies

$$2^{a(b-1)} + 2^{a(b-2)} + \dots + 2^{2a} + 2^a + 1$$

$$\geq 2^a + 1$$

$a \geq 2$

$$\geq 2^2 + 1$$

$$= 5$$

So we have $2^n - 1 = X \cdot Y$ where $X \geq 3$ and $Y \geq 5$.

By HW 1 #7, $2^n - 1$ is not prime ▨