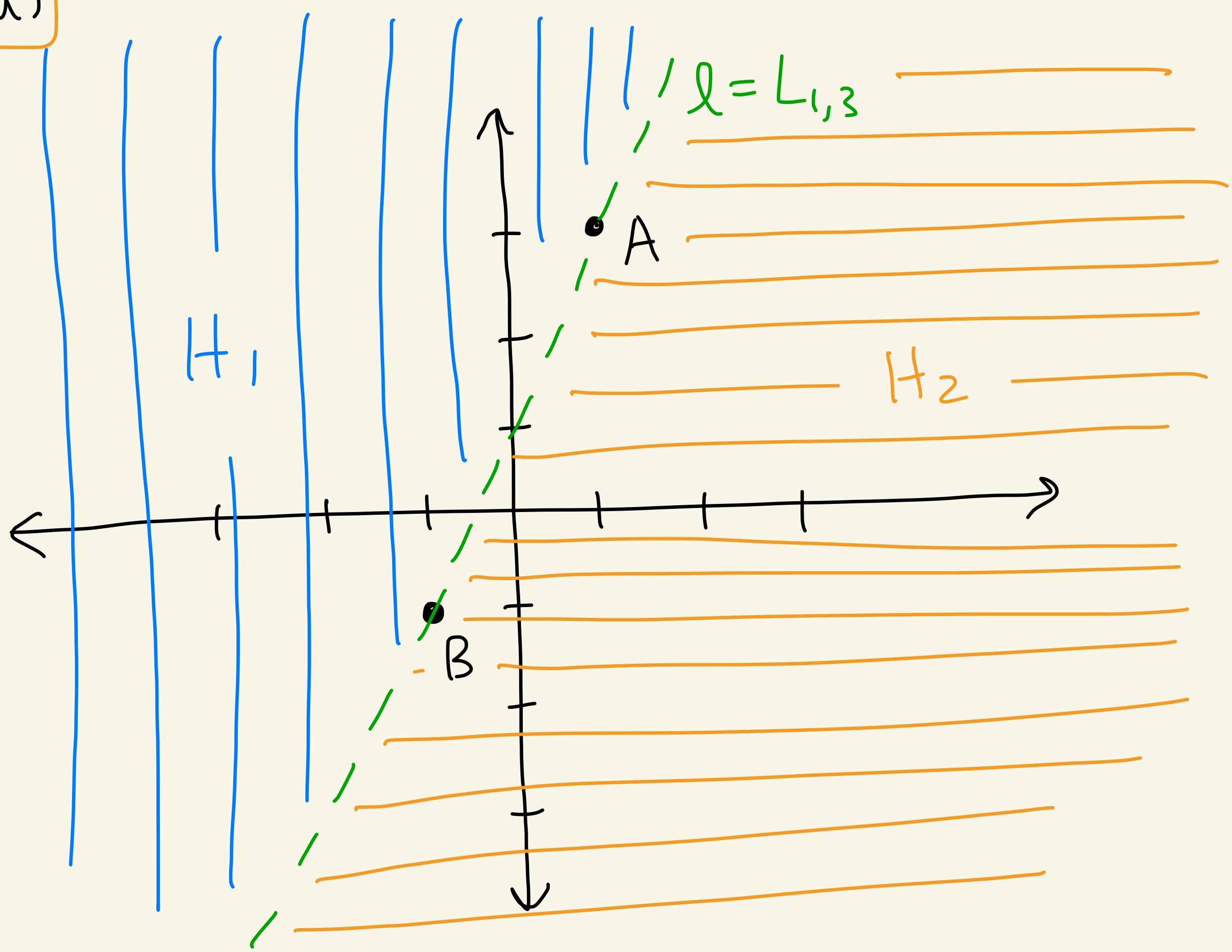
Math 4300
Homework 7
Solutions

① $A=(1,3)$, $B=(-1,1)$

One can show that $\ell = \overleftrightarrow{AB} = L_{1,3}$

(a)



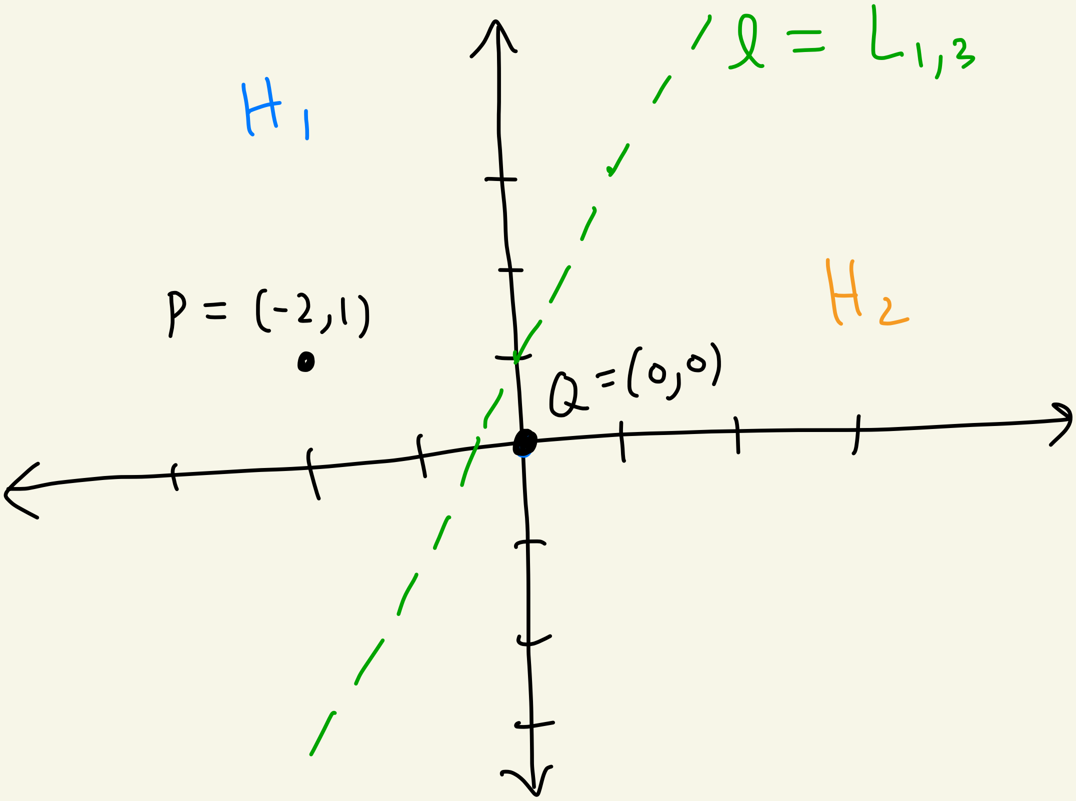
I shaded one half plane orange and the other blue.

The line ℓ is in green.

Note that neither half-plane includes the line ℓ.

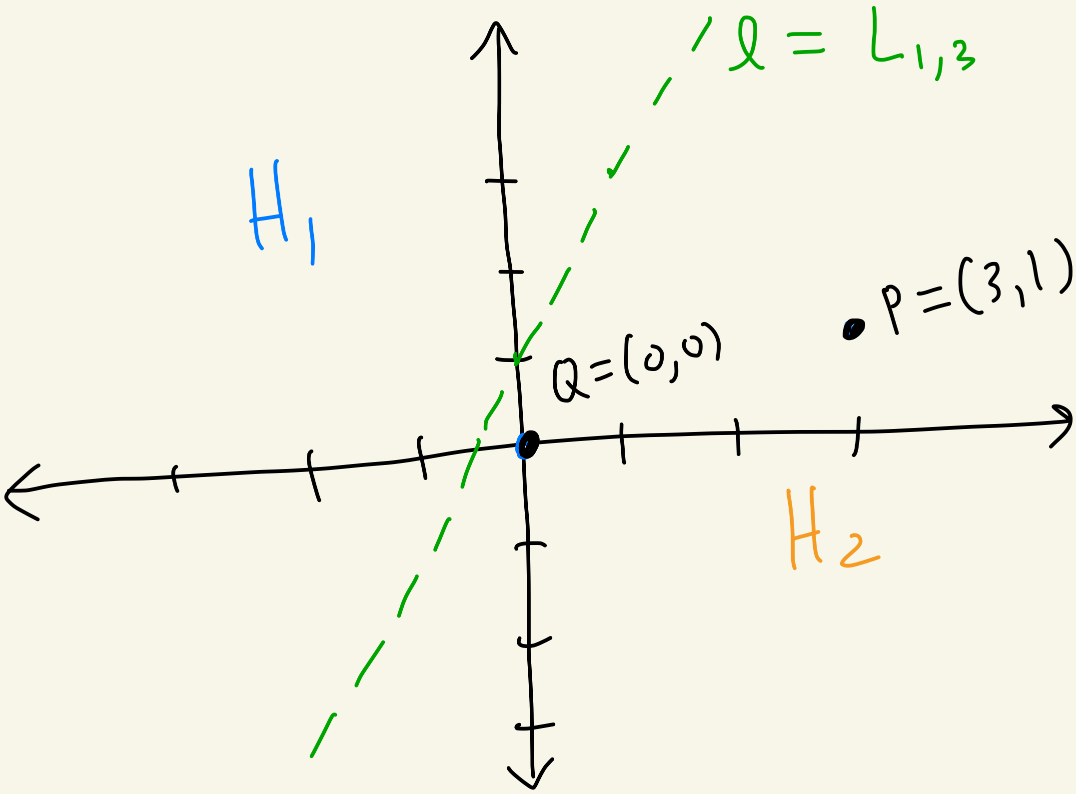
I labelled blue H_1 and orange H_2, but you could reverse them

(b)



P and Q are on opposite sides of $\ell = L_{1,3}$ since $P \in H_1$ and $Q \in H_2$

(c)

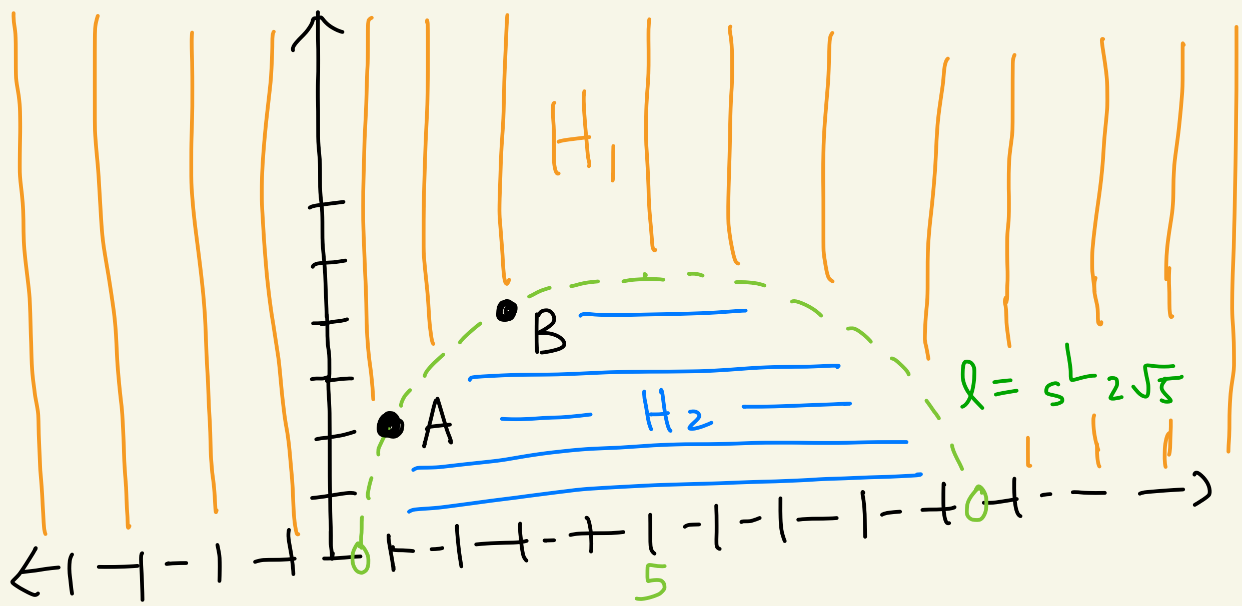


P and Q are on the same side of $\ell = L_{1,3}$ since $P, Q \in H_2$

(2) $A = (1,2)$, $B = (3,4)$

In previous homeworks, we saw that
$\ell = \overleftrightarrow{AB} = {}_5L_{2\sqrt{5}}$ where $2\sqrt{5} \approx 4.47$

(a)



I shaded one half plane orange and the other blue.

The line ℓ is in green.

Note that neither half-plane includes the line ℓ.

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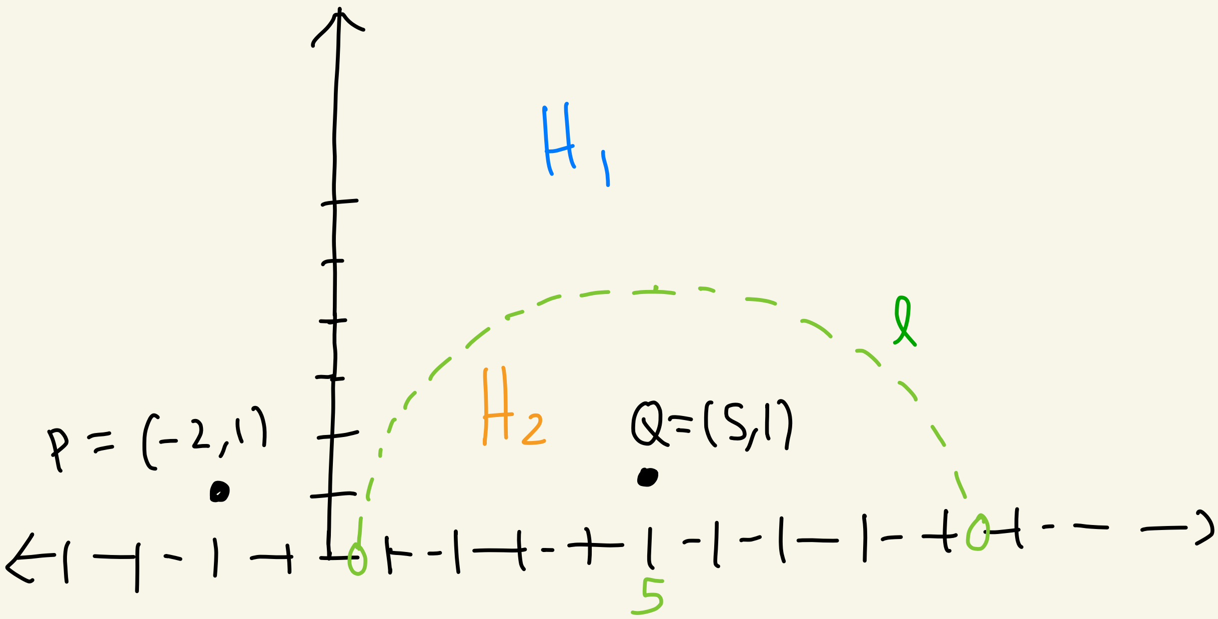
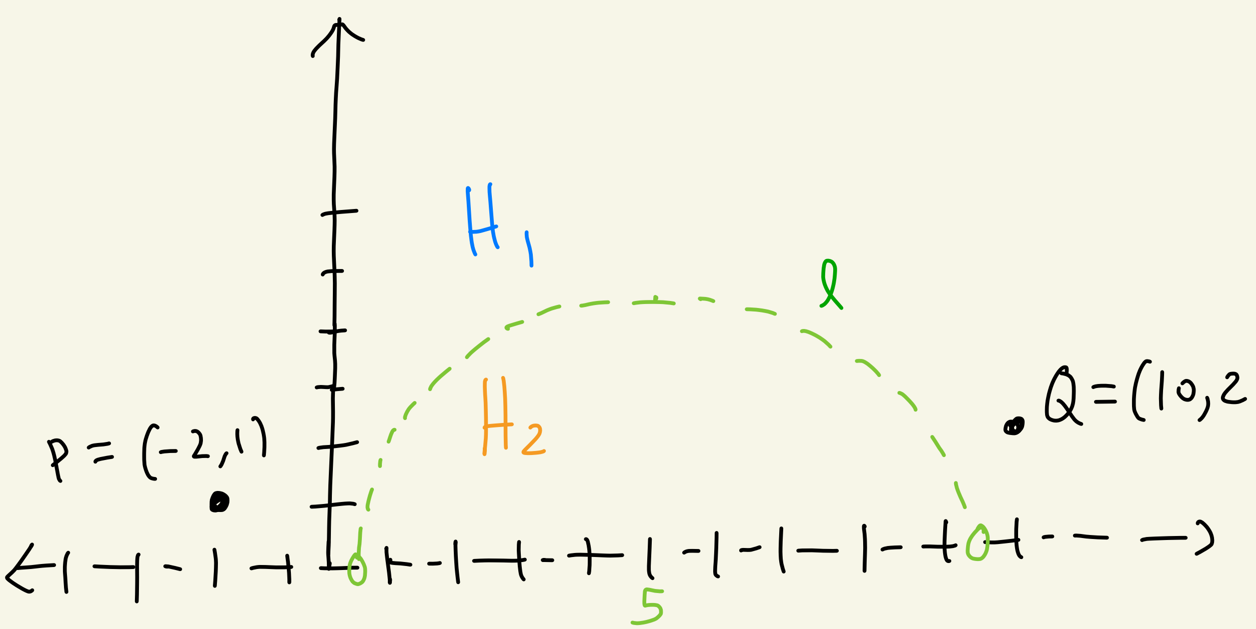
(b)



P and Q are on opposite sides of ℓ.
because $P \in H_1$ while $Q \in H_2$

(c)



P and Q are on the same side of ℓ
because $P \in H_1$ and $Q \in H_1$

③(a) Let $\ell = \overleftrightarrow{XY}$, where $X \neq Y$.

Let $f : \ell \to \mathbb{R}$ be a ruler on ℓ.

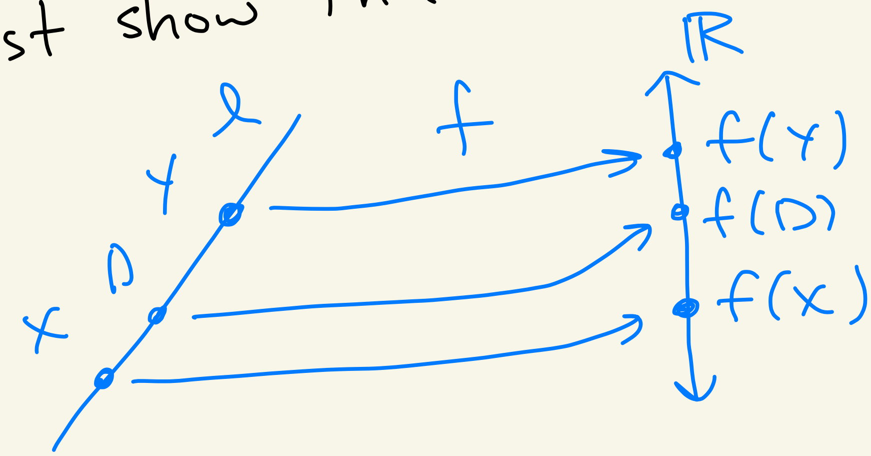
Since $X \neq Y$ we know $f(X) \neq f(Y)$.

So either $f(X) < f(Y)$ or $f(Y) < f(X)$.

<u>case 1:</u> Suppose $f(X) < f(Y)$.

Let $S = \{ D \in \ell \mid f(X) \leq f(D) \leq f(Y) \}$.

We must show that $\overline{XY} = S$.



$\boxed{\overline{XY} \subseteq S:}$

Let $D \in \overline{XY} = \{X, Y\} \cup \{D \in \wp \mid X - D - Y\}$

If $D = X$, then $f(X) = f(D)$ and so $D \in S$.

If $D = Y$, then $f(D) = f(Y)$ and so $D \in S$.

If $X - D - Y$, then by a theorem in class we know $f(X) < f(D) < f(Y)$ and so $D \in S$.

So in all cases, $D \in S$.

Thus, $\overline{XY} \subseteq S$.

$\boxed{S \subseteq \overline{XY}}$: Let $D \in S$.

Then, $f(X) \leq f(D) \leq f(Y)$.

If $f(D) = f(X)$, then since f is one-to-one we must have that $D = X$ and so $D \in \overline{XY}$.

If $f(D) = f(Y)$, then since f is one-to-one we must have that $D = Y$ and so $D \in \overline{XY}$.

If $f(X) < f(D) < f(Y)$, then $X - D - Y$ and so $D \in \overline{XY}$.

These are all the cases thus $S \subseteq \overline{XY}$.

From above $S = \overline{XY}$.

<u>case 2:</u> Suppose $f(Y) < f(X)$.

Let $S = \{ D \in \ell \mid f(Y) \leq f(D) \leq f(X) \}$.

The proof that $S = \overline{XY}$ is similar to the proof of case 1.

Try it if you want more practice.

$\blacksquare$

$\boxed{(3)(b)}$ Let $\ell = \overleftrightarrow{XY}$ where $X \neq Y$.

Let $f : \ell \to \mathbb{R}$ be a ruler.

Since $X \neq Y$ we know $f(X) \neq f(Y)$.

So either $f(X) < f(Y)$ or $f(Y) < f(X)$.

$\boxed{\text{case 1:}}$ Suppose $f(X) < f(Y)$.

Define $g : \ell \to \mathbb{R}$ by

$$g(C) = f(C) - f(X).$$

g is a ruler on ℓ by a thm from class from topic 2

Note that

$$g(X) = f(X) - f(X) = 0$$

and $g(Y) = f(Y) - f(X) > 0$

since $f(Y) > f(X)$ in case 1

Thus, g is a ruler on ℓ with $g(X) = 0$ and $g(Y) > 0$.

we shifted f to make g

By a theorem in class, we have

$$\overrightarrow{XY} = \{ C \in \mathscr{P} \mid 0 \leq g(C) \}$$

Since $g(c) = f(c) - f(x)$

$$\overrightarrow{XY} = \{c \in \mathscr{P} \mid 0 \le f(c) - f(x)\}$$
$$= \{c \in \mathscr{P} \mid f(x) \le f(c)\}$$

$\boxed{\text{Case 2:}}$ Suppose $f(Y) < f(X)$.

Define $g : \ell \to \mathbb{R}$ by
$$g(c) = -(f(c) - f(x))$$

g is a ruler by a thm from class in topic 2

Then, $g(x) = -(f(x) - f(x)) = 0$
and $g(y) = -(f(y) - f(x)) = f(x) - f(y) > 0$.

Then, as in case 1 we will have

thm from class

$$\overrightarrow{XY} = \{c \in \mathscr{P} \mid 0 \le g(c)\}$$
$$= \{c \in \mathscr{P} \mid 0 \le -(f(c) - f(x))\}$$
$$= \{c \in \mathscr{P} \mid 0 \le -f(c) + f(x)\}$$
$$= \{c \in \mathscr{P} \mid f(c) \le f(x)\}$$

$\blacksquare$

④

Let A, B be distinct points in $S \cap T$.

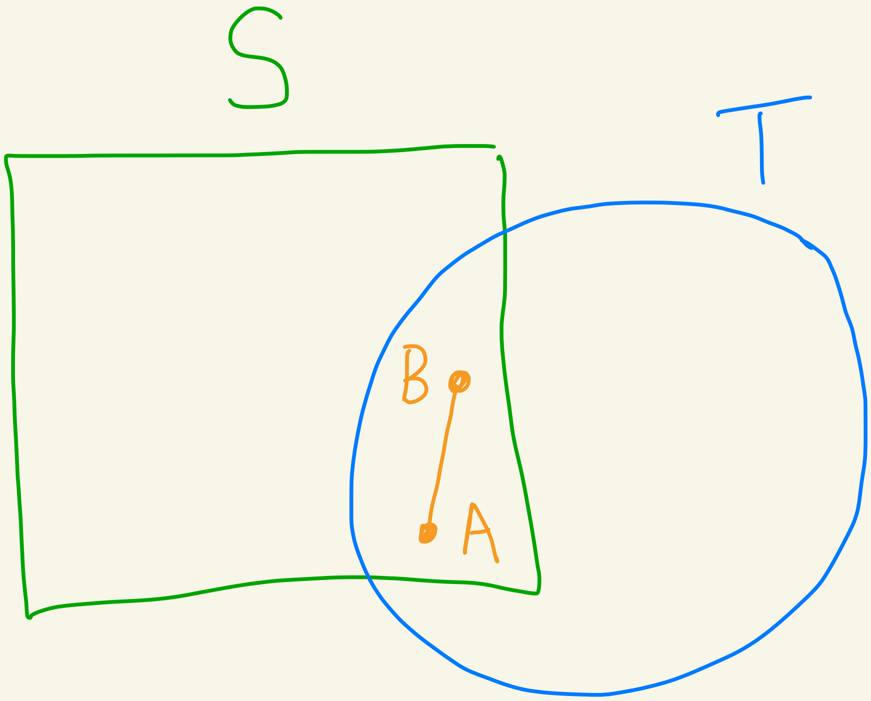
Then $A, B \in S$ and $A, B \in T$.

Since S is convex we have $\overline{AB} \subseteq S$.

Since T is convex we have $\overline{AB} \subseteq T$.

Thus, $\overline{AB} \subseteq S \cap T$.

So, $S \cap T$ is convex.



$\blacksquare$

5(a) ϕ is convex means:

$$(\forall P, Q \in \phi)(\text{If } P \neq Q, \text{ then } \overline{PQ} \subseteq \phi)$$

There are no $P, Q \in \phi$ so this statement is true. $\blacksquare$

5(b) $\{A\}$ is convex means:

$$(\forall P, Q \in \{A\})(\text{If } P \neq Q, \text{ then } \overline{PQ} \subseteq \{A\})$$

There is only the case when $P = A, Q = A$ which makes "If $\underbrace{P \neq Q}_{\text{False}}$, then $\overline{PQ} \subseteq \{A\}$" a true statement (Recall "If F, then —" is always true)

Thus, $\{A\}$ is convex. $\blacksquare$

$\boxed{⑤ (c)}$

Let $P, Q \in \mathscr{P}$.

Then, $\overline{PQ} = \{P, Q\} \cup \{C \in \mathscr{P} \mid P\text{-}C\text{-}Q\}$

By def $\overline{PQ} \subseteq \mathscr{P}$.

So, $\mathscr{P}$ is convex. $\blacksquare$

$\boxed{⑤(d)}$ Let $A, B \in \mathscr{P}$ where $A \neq B$. Let $\ell = \overleftrightarrow{AB}$.
Let $P, Q \in \overline{AB}$ where $P \neq Q$.

$\boxed{\begin{array}{l}\text{\underline{Goal}: We must show that } \overline{PQ} \subseteq \overline{AB}. \\ \text{This will show that } \overline{AB} \text{ is convex.}\end{array}}$

Since $P, Q \in \overline{AB}$ we have $\ell = \overleftrightarrow{AB} = \overleftrightarrow{PQ}$.

Let $f : \ell \to \mathbb{R}$ be a ruler.

Since $A \neq B$ we have
either $f(A) < f(B)$ or $f(B) < f(A)$.

Since $\overline{AB} = \overline{BA}$, we may assume that
$f(A) < f(B)$. Otherwise, just interchange
A and B and relabel them.

Suppose $f(A) < f(B)$.

Since $P, Q \in A, B$ from problem 3 of this
homework we have $f(A) \leq f(P) \leq f(B)$
and $f(A) \leq f(Q) \leq f(B)$. $(*)$

Now break this into 2 cases.

If $f(P) < f(Q)$, then

part (a)

$$\overline{PQ} = \{C \in \mathscr{P} \mid f(P) \le f(C) \le f(Q)\}$$

$$\subseteq \{C \in \mathscr{P} \mid f(A) \le f(C) \le f(B)\} = \overline{AB}$$

if $f(P) \le f(C) \le f(Q)$,

then $f(A) \overset{(*)}{\le} f(P) \le f(C) \le f(Q) \overset{(*)}{\le} f(B)$

If $f(Q) < f(P)$, then

part (a)

$$\overline{PQ} = \{C \in \mathscr{P} \mid f(Q) \le f(C) \le f(P)\}$$

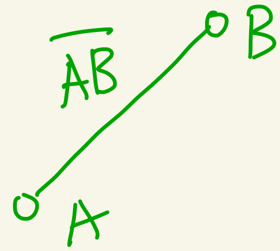
$$\subseteq \{C \in \mathscr{P} \mid f(A) \le f(C) \le f(B)\} = \overline{AB}$$

if $f(Q) \le f(C) \le f(P)$,

then $f(A) \overset{(*)}{\le} f(Q) \le f(C) \le f(P) \overset{(*)}{\le} f(B)$

In both cases, $\overline{PQ} \subseteq \overline{AB}$. Thus, $\overline{AB}$ is convex. ∎

$\boxed{⑤(e)}$ Let A, B be distinct points.



We want to show that
$$\text{int}(\overline{AB}) = \overline{AB} - \{A, B\}$$
is convex.

Let $\ell = \overleftrightarrow{AB}$.

Let $f: \ell \to \mathbb{R}$ be a ruler.

We can have $f(A) < f(B)$ or $f(B) < f(A)$.

Since $\text{int}(\overline{AB}) = \overline{AB} - \{A, B\} = \overline{BA} - \{A, B\} = \text{int}(\overline{BA})$

we may assume that $f(A) < f(B)$

Otherwise, just interchange and relabel A and B.

Thus, assume $\boxed{f(A) < f(B).}$

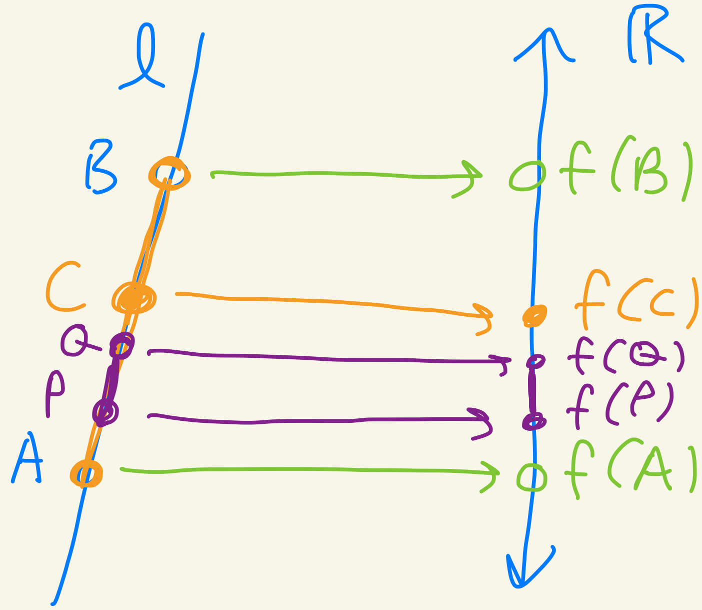
From problem 3 we have
$$\overline{AB} = \{ C \in \mathscr{P} \mid f(A) \leq f(C) \leq f(B) \}$$

Since $\text{int}(\overline{AB}) = \overline{AB} - \{A, B\}$ we get
$$\text{int}(\overline{AB}) = \{ C \in \mathscr{P} \mid f(A) < f(C) < f(B) \}.$$

Let $P, Q \in \operatorname{int}(\overline{AB})$.

We must show that $\overline{PQ} \subseteq \operatorname{int}(\overline{AB})$.

This will show that $\operatorname{int}(\overline{AB})$ is convex.



Since $P, Q \in \operatorname{int}(\overline{AB})$ we know that $f(A) < f(P) < f(B)$ and $f(A) < f(Q) < f(B)$

If $f(P) < f(Q)$, then

problem 3

$$\overline{PQ} = \{ C \in \mathscr{P} \mid f(P) \leq f(C) \leq f(Q) \}$$

$$= \{ C \in \mathscr{P} \mid f(A) < f(C) < f(B) \}$$

since $f(A) < f(P)$ and $f(Q) < f(B)$

$$= \operatorname{int}(\overline{AB}).$$

So, $\overline{PQ} \subseteq \operatorname{int}(\overline{AB})$.

If $f(Q) < f(P)$, then

$$\begin{aligned}
\overline{PQ} &\overset{\text{problem 3}}{=} \{ C \in \mathscr{P} \mid f(Q) \leq f(C) \leq f(P) \} \\
&= \{ C \in \mathscr{P} \mid f(A) < f(C) < f(B) \}
\end{aligned}$$

since $f(A) < f(Q)$ and $f(P) < f(B)$

$$= \operatorname{int}(\overline{AB}).$$

So, $\overline{PQ} \subseteq \operatorname{int}(\overline{AB})$.

In either case $\overline{PQ} \subseteq \operatorname{int}(\overline{AB})$.

So, $\operatorname{int}(\overline{AB})$ is convex.

∎

$\boxed{(5)(f)}$ Let $P, Q \in \overleftrightarrow{AB}$.

Then $\overleftrightarrow{PQ} = \overleftrightarrow{AB}$.

Thus, $\overline{PQ} \subseteq \overleftrightarrow{PQ} = \overleftrightarrow{AB}$.

So, $\overleftrightarrow{AB}$ is convex. $\blacksquare$

(5)(g) Let A, B be distinct points.

Let $f: \ell \to \mathbb{R}$ be a ruler on $\ell = \overleftrightarrow{AB}$ where $f(A) = 0$ and $f(B) > 0$.

Then

$$\overrightarrow{AB} = \{ C \in \ell \mid 0 \leq f(C) \}$$

thm from class

Let $P, Q \in \overrightarrow{AB}$ be distinct points.

We must show that $\overline{PQ} \subseteq \overrightarrow{AB}$.

Since $P, Q \in \overrightarrow{AB}$ we know that $0 \leq f(P)$ and $0 \leq f(Q)$.

<u>case 1:</u> Suppose $f(Q) < f(P)$

We have that

$$\overline{PQ} \overset{\text{problem 3}}{=} \{ C \in \ell \mid \underbrace{f(Q)}_{0 \leq f(Q)} \leq f(C) \leq f(P) \}$$

$$\subseteq \{ C \in \ell \mid 0 \leq f(C) \} = \overrightarrow{AB}$$

So we get $\overline{PQ} \subseteq \overrightarrow{AB}$.

<u>Case 2:</u> Suppose $f(P) < f(Q)$.

problem 3

Then,

$$\overline{PQ} = \{ C \in \mathscr{P} \mid \underbrace{f(P)}_{0 \leq f(P)} \leq f(C) \leq f(Q) \}$$

$$\subseteq \{ C \in \mathscr{P} \mid 0 \leq f(C) \} = \overrightarrow{AB}$$

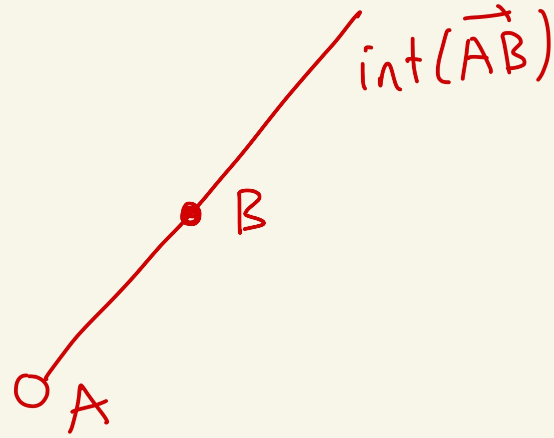
So we get $\overline{PQ} \subseteq \overrightarrow{AB}$.

In either case we get that $\overline{PQ} \subseteq \overrightarrow{AB}$.

So, $\overrightarrow{AB}$ is convex. ∎

5 (h) Let A, B be distinct points.

We want to show that
$$\text{int}(\overrightarrow{AB}) = \overrightarrow{AB} - \{A\}$$
is convex.



Let $l = \overleftrightarrow{AB}$.

Let $f: l \to \mathbb{R}$ be a ruler where $f(A) = 0$ and $f(B) > 0$.

Then, from class we have that
$$\overrightarrow{AB} = \{ C \in \mathscr{P} \mid 0 \leq f(C) \}$$

Since $f(A) = 0$ and f is one-to-one this gives us that
$$\boxed{\text{int}(\overrightarrow{AB}) = \{ C \in \mathscr{P} \mid 0 < f(C) \}}.$$

Let $P, Q \in \text{int}(\overrightarrow{AB})$.

Goal: We must show that $\overline{PQ} \subseteq \text{int}(\overrightarrow{AB})$ to show that $\text{int}(\overrightarrow{AB})$ is convex.

Then, from above we have that

$$0 < f(P) \text{ and } 0 < f(Q).$$

$\boxed{\text{Case 1:}}$ Suppose $0 < f(P) < f(Q)$.

problem 3

Then, we have

$$\overline{PQ} = \{ C \in \mathcal{P} \mid f(P) \le f(C) \le f(Q) \}$$

$$\subseteq \{ C \in \mathcal{P} \mid 0 \le f(C) \} = \operatorname{int}(\overrightarrow{AB})$$

$\boxed{\text{since } 0 < f(P)}$

So, $\overline{PQ} \subseteq \operatorname{int}(\overrightarrow{AB})$.

$\boxed{\text{Case 2:}}$ Suppose $0 < f(Q) < f(P)$.

Then, we have that

problem 3

$$\overline{PQ} = \{ C \in \mathscr{P} \mid f(Q) \le f(C) \le f(P) \}$$

$$\subseteq \{ C \in \mathscr{P} \mid 0 \le f(C) \} = \operatorname{int}(\overrightarrow{AB})$$

since $0 < f(Q)$

So, $\overline{PQ} \subseteq \operatorname{int}(\overrightarrow{AB})$.

In both cases $\overline{PQ} \subseteq \operatorname{int}(\overrightarrow{AB})$.

So, $\operatorname{int}(\overrightarrow{AB})$ is convex.

6 (a)

Let ℓ be a line and $P \notin \ell, Q \notin \ell$.
Let H_1, H_2 be the half-planes determined by ℓ.

($\Rightarrow$) Suppose P and Q lie on opposite sides of ℓ.

If $P \in H_1$ and $Q \in H_2$, the by PSA (iv) we have $\overline{PQ} \cap \ell \neq \phi$

If $P \in H_2$ and $Q \in H_1$, then by PSA (iv) we have $\overline{PQ} \cap \ell \neq \phi$.

($\Leftarrow$) Suppose $\overline{PQ} \cap \ell \neq \phi$.
Why must P and Q lie on opposite sides of ℓ?
Suppose they didn't, ie they were on the same side of ℓ.
Without loss of generality, assume $P, Q \in H_1$.
H_1 is convex so $\overline{PQ} \subseteq H_1$
But $\ell \cap H_1 = \phi$
So, if $\overline{PQ} \subseteq H_1$ then $\overline{PQ} \cap \ell = \phi$ which is a contradiction.

Thus, P and Q lie on opposite sides of ℓ. $\boxed{//}$

6 (b)

Let ℓ be a line and $P \notin \ell, Q \notin \ell$.
Let H_1, H_2 be the half-planes detemined by ℓ.

($\Rightarrow$) Suppose P and Q lie on the same side of ℓ.

Without loss of generality, suppose $P, Q \in H_1$.
Since H_1 is convex we have $\overline{PQ} \subseteq H_1$
Since $H_1 \cap \ell = \phi$ and $\overline{PQ} \subseteq H_1$
we know $\overline{PQ} \cap \ell = \phi$.

($\Leftarrow$) Suppose $\overline{PQ} \cap \ell = \phi$.
We want to show that P, Q lie on the same side of ℓ.

Suppose they lie on opposite sides of ℓ.
Suppose $P \in H_1$ and $Q \in H_2$, then by PSA (iv) we would have $\overline{PQ} \cap \ell \neq \phi$ which isn't the case.

Same thing if $P \in H_2$ and $Q \in H_1$
Thus, P, Q lie on the same side of ℓ. $\blacksquare$

(7) Let H_1, H_2 be the half-planes determined by ℓ.

Suppose P and Q are on opposite sides of ℓ, and Q and R are on opposite sides of ℓ.

Since P and Q lie on opposite sides of ℓ then either (i) $P \in H_1$ and $Q \in H_2$ or (ii) $P \in H_2$ and $Q \in H_1$.

<u>Case 1:</u> Suppose $P \in H_1$ and $Q \in H_2$.

Since $Q \in H_2$ and Q and R lie on opposite sides of ℓ we must have $R \in H_1$.

Thus, $P \in H_1$ and $R \in H_1$.

So, P and R lie on the same side of ℓ.

<u>Case 2:</u> Suppose $P \in H_2$ and $Q \in H_1$.

Since $Q \in H_1$ and Q and R lie on opposite sides of ℓ we must have $R \in H_2$.

Thus, $P \in H_2$ and $R \in H_2$.

So, P and R lie on the same side of ℓ.

$\blacksquare$

8. Let H_1, H_2 be the half-planes determined by ℓ.

Suppose P and Q are on opposite sides of ℓ, and Q and R are on the same side of ℓ.

Since P and Q lie on opposite sides of ℓ then either (i) $P \in H_1$ and $Q \in H_2$ or (ii) $P \in H_2$ and $Q \in H_1$.

Case 1: Suppose $P \in H_1$ and $Q \in H_2$.
Since $Q \in H_2$ and Q and R lie on the same side of ℓ we must have $R \in H_2$.
Thus, $P \in H_1$ and $R \in H_2$
so, P and R lie on opposite sides of ℓ.

Case 2: Suppose $P \in H_2$ and $Q \in H_1$.
Since $Q \in H_1$ and Q and R lie on the same side of ℓ we must have $R \in H_1$.
Thus, $P \in H_2$ and $R \in H_1$.
so, P and R lie on opposite sides of ℓ.

$\blacksquare$