Math 4300 Homework #5 Solutions









3
$$A = (1,2)$$
, $B = (3,4)$ in
the hyperbolic plane
What is the line that $A_{J}B$ lie on?
Plug A and B into $(x-c)^{2} + y^{2} = r^{2}$ to get
 $(1-c)^{2} + 2^{2} = r^{2}$ (1)
 $(3-c)^{2} + y^{2} = r^{2}$ (2)
 $-2c + c^{2} + 5 = r^{2}$ (2)
 $-6c + c^{2} + 25 = r^{2}$ (3)
 $-6c + c^{2} + 25 = r^{2}$ (4)
 $-2c + c^{2} + 25 = r^{2}$ (5)
 $-6c + c^{2} + 25 = r^{2}$ (7)
 $-6c + c^{2} + 25 = r^{2}$ (7)
 $-6c + c^{2} + 25 = r^{2}$ (7)
 $-6c + c^{2} + 25 = r^{2}$ (8)
 $-6c + c^{2} + 25 = r^{2}$ (9)
 $-6c + c^{2} + 25 = r^{2}$ (10)
 $-7c + 25 = r^{2}$ (10)
 $-7c$







 (\mathcal{A})



So,

$$(x,y) = C = (0,0) + t(2-0, |-0| = (zt, t)$$

That is, $X = 2t, y = t$

Also since
$$d(A, c) = \sqrt{20}$$
 we know
 $\sqrt{(x-0)^2 + (y-0)^2} = \sqrt{20}$
 $d(A, c)$
This gives $x^2 + y^2 = 20$
Plug $x = 2t, y = t$ into $x^2 + y^2 = 20$
to get $4t^2 + t^2 = 20$.
So, $t^2 = \frac{20}{5} = 4$. Thus, $t = \pm 2$.
Since $t \ge 0$ must be
true from above
we get $t = 2$
Thus, $C = (2t, t)$
 $= (4, 21)$.
Then $C \in AB$ and
 $AC = d(A, c) = \sqrt{20} = d(P, Q) = PQ$

5) P = (1,2), Q = (1,4)A = (0,2), B = (1, J3) Let us first determine the liner PQ and AB PQ = L since P,Q both have x-coordinate 1. For \overrightarrow{AB} we plug them into $(x-c)^2 + y^2 = r^2$ to get $\begin{pmatrix} 0-c \end{pmatrix}^{2} + 2^{2} = r^{2} \\ (1-c)^{2} + \sqrt{3}^{2} = r^{2} \end{pmatrix} \longrightarrow \begin{pmatrix} c^{2} + 4 = r^{2} \\ -2c + c^{2} + 4 = r^{2} \end{pmatrix}$ (1-2) gives 2c=0 or c=0. Then \bigcirc gives $r = \int \delta^2 f 4 = 2$. Q=(1,4) So, $\overrightarrow{AB} = c L_r = o L_2$. P = (1,2)A = (0, 2)For picture: $B = (1, \sqrt{3})$ √3≈1.73 AB

We now need

$$PQ = d_{\mu}(P,Q) = \left| \ln\left(\frac{z}{4}\right) \right|$$

$$= \left| \ln\left(\frac{z}{4}\right) \right|$$

$$= \left| \ln\left(\frac{z}{2}\right) \right| = -\ln\left(\frac{t}{2}\right) \approx 0.69$$

$$\ln\left(\frac{t}{2}\right) \approx -0.69$$
We want to find $C \in AB$ where $AC = -\ln\left(\frac{t}{2}\right)$.
Let $C = (x,y)$.
Since $C \in AB$ we know $x^{2} + y^{2} = 2$ and $x \ge 0$
Since $C \in AB$ we know $x^{2} + y^{2} = 2$ and $x \ge 0$
Since $C \in AB$ we know $x^{2} + y^{2} = 2$ and $x \ge 0$
Since $AC = -\ln\left(\frac{t}{2}\right)$ we get

$$= \ln\left(\frac{t}{2}\right) = AC = d_{\mu}\left((0,2),(x,y)\right)$$

$$= \left| \ln\left(\frac{(0-0+2)/2}{(x-0+2)/y}\right) \right|$$

$$= \left| \ln\left(\frac{1}{(x+2)}\right) \right|$$

So we need
$$|n(\frac{4}{xt^2})| = \pm (-|n(\frac{1}{2})| = \pm \ln(\frac{1}{2})$$

That is, $|n(\frac{4}{xt^2}) = \ln(\frac{1}{2})$ or $|n(\frac{4}{xt^2}) = -|n(\frac{1}{2}) = |n(2)$
So we get either
 $\frac{4}{xt^2} = \frac{1}{2}$ or $\frac{4}{xt^2} = 2$
So either
 $2y = x + 2$ or $y = 2x + 4$
or $y = \frac{1}{2}x + 1$ or $y = 2x + 4$
Plug this into $x^2 + y^2 = 2$ to get either
 $x^2 + (\frac{1}{2}x + 1)^2 = 2$
or $x^2 + (2x + 4)^2 = 2$
 $y = \frac{5}{2}x^2 + 4x - 1 = 0$
 $y = \frac{5}{2}x^2 + 4x - 4 = 0$
 $y = \frac{5}{2}x^2 + 4x - 4 = 0$
 $y = \frac{5}{2}x^2 + 4x - 4 = 0$
 $y = \frac{5}{2}x^2 + 4x - 4 = 0$
 $y = \frac{5}{2}x^2 + 4x - 4 = 0$
 $y = \frac{5}{2}x^2 + 4x - 4 = 0$
 $y = \frac{5}{2}x^2 + 4x - 4 = 0$
 $y = \frac{5}{2}x^2 + 4x - 4 = 0$
 $y = \frac{5}{2}x^2 + 4x - 4 = 0$
 $y = \frac{5}{2}x^2 + 4x - 4 = 0$
 $y = \frac{5}{2}x^2 + 4x - 4 = 0$
 $y = \frac{5}{2}x^2 + 4x - 4 = 0$
 $y = \frac{5}{2}x^2 + 4x - 4 = 0$
 $y = \frac{5}{2}x^2 + 4x - 4 = 0$
 $y = \frac{5}{2}x^2 + 4x - 4 = 0$
 $y = \frac{5}{2}x^2 + 4x - 4 = 0$
 $y = \frac{5}{2}x^2 + 4x - 4 = 0$
 $y = \frac{5}{2}x^2 + 4x - 4 = 0$
 $y = \frac{5}{2}x^2 + 4x - 4 = 0$
 $y = \frac{5}{2}x^2 + 4x - 4 = 0$
 $y = \frac{5}{2}x^2 + 4x - 4 = 0$
 $y = \frac{5}{2}x^2 + 4x - 4 = 0$
 $y = \frac{5}{2}x^2 + 4x - 4 = 0$
 $y = \frac{5}{2}x^2 + 4x - 4 = 0$
 $y = \frac{5}{2}x^2 + 4x - 4 = 0$
 $y = \frac{5}{2}x^2 + 4x - 4 = 0$
 $y = \frac{5}{2}x^2 + 4x - 4 = 0$
 $y = \frac{5}{2}x^2 + 4x - 4 = 0$
 $y = \frac{5}{2}x^2 + 4x - 4 = 0$
 $y = \frac{5}{2}x^2 + 4x - 4 = 0$
 $y = \frac{5}{2}x^2 + 4x - 4 = 0$
 $y = \frac{5}{2}x^2 + 4x - 4 = 0$
 $y = \frac{5}{2}x^2 + 4x - 4 = 0$
 $y = \frac{5}{2}x^2 + 4x - 4 = 0$
 $y = \frac{5}{2}x^2 + 4x - 4 = 0$
 $y = \frac{5}{2}x^2 + 4x - 4 = 0$

(1) gives
$$x = -\frac{4 \pm \sqrt{4^2 - 4(s)(-4)}}{2(s)} = -\frac{4 \pm \sqrt{96}}{10}$$

 $= -\frac{2}{5} \pm \frac{\sqrt{96}}{10}, -\frac{2}{5} - \frac{\sqrt{96}}{10}$
 ≈ 0.5798 negative
Since we need $x \ge 0$ this
Case gives $x = -\frac{2}{5} \pm \frac{\sqrt{96}}{10}$
and $y = \frac{1}{2}x \pm 1 = -\frac{1}{5} \pm \frac{\sqrt{96}}{20} \pm 1 = \frac{4}{5} \pm \frac{\sqrt{96}}{20} \approx 1.2899$.

2) gives
$$X = \frac{-16 \pm 0.16^{2} - 4(5)(14)}{2(51)} = \frac{-16 \pm 0.24}{10}$$

But this is a complex number.

But this is a complex number.
Thus,
$$C = \left(\frac{-2}{5} + \frac{\sqrt{96}}{10} \right), \frac{4}{5} + \frac{\sqrt{96}}{20} \right) \approx \left(0.5798, 1.2899 \right)$$

וור



$$\overline{AB} = \overline{AB} \cup \{ c \in \mathcal{P} \mid A - B - C \}$$

$$\overline{AB}$$

$$= \{ \overline{A, B} \} \cup \{ c \in \mathcal{P} \mid A - C - B \}$$

$$\cup \{ c \in \mathcal{P} \mid A - B - C \}$$

$$= AB$$

Thus, $\overrightarrow{AB} \subseteq \overrightarrow{AB}$.

(c)
$$\overrightarrow{AB} = \overrightarrow{AB} \cup \{ c \in 9 | A - B - C \}$$

and $\overrightarrow{BA} = \overrightarrow{BA} \cup \{ c \in 9 | c - B - A \}$
 $= \overrightarrow{AB} \cup \{ c \in 9 | c - B - A \}$
from part (A)
Thus, $\overrightarrow{AB} \in \overrightarrow{AB}$ and $\overrightarrow{AB} \in \overrightarrow{BA}$.
So, $\overrightarrow{AB} \in \overrightarrow{AB} \cap \overrightarrow{BA}$.
Now let's show $\overrightarrow{AB} \cap \overrightarrow{BA} \leq \overrightarrow{AB}$.
Now let's show $\overrightarrow{AB} \cap \overrightarrow{BA} \leq \overrightarrow{AB}$.
Now let's show $\overrightarrow{AB} \cap \overrightarrow{BA} \leq \overrightarrow{AB}$.
Now let's show $\overrightarrow{AB} \cap \overrightarrow{BA} \leq \overrightarrow{AB}$.
Now let's show $\overrightarrow{AB} \cap \overrightarrow{BA} \leq \overrightarrow{AB}$.
Now let's show $\overrightarrow{AB} \cap \overrightarrow{BA} \leq \overrightarrow{AB}$.
Now let's show $\overrightarrow{AB} \cap \overrightarrow{BA} \leq \overrightarrow{AB}$.
Now let's show $\overrightarrow{AB} \cap \overrightarrow{BA} \leq \overrightarrow{AB}$.
Now let's show $\overrightarrow{AB} \cap \overrightarrow{BA} \leq \overrightarrow{AB}$.
So ($\overrightarrow{C} \in \overrightarrow{AB}$ either $(\overrightarrow{C} \in \overrightarrow{BA} = \overrightarrow{AB} \text{ or } \overrightarrow{B} - \overrightarrow{C})$ (#)
Since $c \in \overrightarrow{BA}$ either $(\overrightarrow{C} \in \overrightarrow{AB} = \overrightarrow{AB} \text{ or } \overrightarrow{B} - \overrightarrow{C})$ (#)
Since $c \in \overrightarrow{AB}$ then we are done.

Suppose A-B-C.
From (**) we get either CEAB or B-A-C
If CEAB then we are done.
Suppose B-A-C.
Then we would have A-B-C and B-A-C.
Then we would have Know A-B-C
But from HW 4 we know A-B-C
But from HW 4 we know A-B-C
Ind B-A-C cannot both happen
and B-A-C cannot both happen
Thus this case cannot occur.
Thus this case cannot occur.
Thus, CEAB.
So, ABABASA.
Thuefore from the above we get
that
$$\overline{AB} = \overline{AB} \cap \overline{BA}$$
.



L

-

(B) Suppose that A-B-C and P-Q-R
and
$$\overline{AB} \simeq \overline{PQ}$$
 and $\overline{AC} \simeq \overline{PR}$.
(Goal: We must show that $\overline{BC} \simeq \overline{QR}$.
Since A-B-C and P-Q-R we know
 $d(A,B)+d(B,C) = d(A,C)$
 $d(P,Q)+d(Q,R) = d(P,R)$.
Since $\overline{AB} \simeq \overline{PQ}$ and $\overline{AC} \simeq \overline{P/2}$
We know that
 $d(A,B) = d(P,Q)$
 $d(A,B) = d(P,Q)$
 $and d(A,C) = d(P,R)$.
Thus,
 $d(B,C) = d(A,C) - d(A,B)$
 $(\overset{(*)}{=} d(Q,R)$.
Sue $\overline{BC} \simeq \overline{QR}$.

)

9(a) Let
$$A,B,C \in \mathcal{P}$$
 with $A \neq B$.
Let $C \in \overline{AB}$ and $C \neq A$.
We must show that $\overline{AB} = \overline{AC}$.
Let $I = \overline{AB}$.
Let f be a ruler on I where
 $f(A) = 0$ and $f(B) > 0$.
Then from class we
have that
 $\overline{AB} = \{E \in \mathcal{P} \mid 0 \leq f(E)\}$.
 $A = \{F(A) = 0\}$ of $f(B) = 0$
Since $C \in \overline{AB}$ we
know $0 \leq \overline{f(C)}$.
Since $C \neq A$ and $f(A) = 0$ and f
Since $C \neq A$ and $f(A) = 0$ and f .
So in fact $0 < \overline{f(C)}$.

Thus, f(A) = 0 and f(C) > 0. Since l = AB = AC again from the theorem in class we get $\overline{AC} = \{ E \in \mathcal{P} \mid 0 \le f(E) \}$. Thus, $\overline{AB} = \{ E \in \mathcal{P} \mid 0 \le f(E) \} = \overline{AC}$.

(9)(b) Suppose
$$\overrightarrow{AB} = \overrightarrow{CD}$$
.
We must show that $A = C$.
Suppose that $A \neq C$.
Then, from part(a), since $C \in \overrightarrow{AB}$ and $C \neq A$
we get $\overrightarrow{AC} = \overrightarrow{AB}$.
So, $\overrightarrow{AC} = \overrightarrow{AB} = \overrightarrow{CD}$.
Thus, A, B, C, D all lie on $l = \overrightarrow{AC}$.
Let $f: l \rightarrow \mathbb{R}$ be a ruler centered
Let $f: l \rightarrow \mathbb{R}$ be a ruler centered
at A where $f(A) = O$ and $f(C) > O$.
From class this ruler satisfies
 $l = \overrightarrow{AC} = \{X \in \overrightarrow{AC} \mid f(X) \ge O\}$
So, $f(A) < f(C)$.
We want to show that $f(C) < f(D)$.
Suppose instead that $f(A) < f(D) < f(C)$.

Let
$$g: l \rightarrow IR$$
 be
given by $g(X) = -(f(X) - f(C))$
From Topic 2 lectures we know that
that g is a vuler on l .
Note that $g(C) = -(f(C) - f(C)) = 0$
and $g(D) = -(f(D) - f(C))$
 $= f(C) - f(D) > 0$
Since g is a vuler on l
where $g(C) = 0$ and $g(D) > 0$,
from
that $CD = \{X \mid g(X) > 0\}$.
Since f is onto R there exists
 $E \in \mathcal{P}$ where $f(C) < f(E)$
Fir example pick E where
 $f(E) = f(C) + 1$

Then we have f(A) < f(D) < f(c) < f(E).Since f(E)>f(c)>f(A)=0 we know that EEAC However, g(E) = -(f(E) - f(C))= f(C) - f(E) < 0Thus, E&CD. Since CD= {x1g(x1>o} Therefore, EEAC and EECD. But Ac=CD. So no such E exists. Contractiction. Therefore, f(c) < f(D), $S_{0}, f(A) < f(c) < f(0).$

Let
$$h: l \rightarrow R$$
 be given by
 $h(x) = f(x) - f(c)$
By topic 2, h is a ruler on l.
Then, $h(c) = f(c) - f(c) = 0$
and $h(D) = f(D) - f(c) > D$
So,
 $c\overline{D} = \{x \in \mathcal{P}\} h(x) > 0\}$
But, $h(A) = f(A) - f(c) < 0$
So, $A \notin C\overline{D}$.
However $A \in AB$ and $\overline{AB} = \overline{CD}$.
Contradiction.
Thus, the uriginal assumption that $A \neq C$
and $A = C$.

(10)(a) $S = \{ C \in \mathbb{R}^2 | C = A + t(B - A) \text{ where } 0 \leq t \leq l \}.$ Let We must show that $\overline{AB} = S$. $\overline{AB} \leq S$: Then either C=A or C=B or A-C-B. If C = A, then set t = 0 and we ge+C=A=A+O(B-A).So, if C=A, then CES. If C=B, then set t=1 and we get $C = B = A + I \cdot (B - A)$ So if C=B, then CES. By HW 4 there exists t with O<t< | Suppose A-C-B. where C = A + t(B - A). Then in this case also we have that CES.

 $S_{o}, \overline{AB} \leq S.$ SSAB : Suppose CES. Then, C = A + t(B - A) where $0 \le t \le l$. If t=0, then $C = A+O(B-A) = A \in \overline{AB}$ If t=l, then $C = A + I \cdot (B - A) = B \in \overline{AB}$. Then by HW 4, we have A-C-B. So in this last case CEAB. Thus, SEAB.

Since ABES and SSAB we have that AB=S.

(lo(b) Let $S = \{ C \in \mathcal{B} \mid C = A + t(B - A) \text{ where } 0 \leq t \}.$ We must show that $\overrightarrow{AB} = S$. We prove $\overrightarrow{AB} \subseteq \overrightarrow{S}$ and $\overrightarrow{S} \subseteq \overrightarrow{AB}$. (ABSS): Let CEAB. So, from class we know that (+) C = A + t(B-A) where $t \in \mathbb{R}$. Then, CEAB. We must show that $t \ge 0$. Suppose instead that t < 0. We show this leads to a contradiction. Casel: Suppose ÂB=L1 for some JER. Let $A = (d, y_n), B = (d, y_b), C = (d, y_c)$ Su, (*1 gives (d, yc) = (d, ya+ t(yb- ya))

Let
$$f: L_{1} \rightarrow IR$$
 be the standard
ruler, that is $f(d,y) = Y$.
Then applying f to the above gives
 $Y_{c} = Y_{n} + t (Y_{b} - Y_{a})$.
 f gives $Vs two options$:
 $Either f(A) < f(B)$
or $f(B) < f(A)$.
That is either
 $Y_{n} < Y_{b}$
or $Y_{b} < Y_{a}$.
(ase $|(i)$: Suppose $Y_{a} < Y_{b}$.
Then, $Y_{c} - Y_{a} = t (Y_{b} - Y_{a}) < 0$
 $< 0 > 0$

$$So, Y_c < Y_a$$
.

Then
$$y_c < y_a < y_b$$
.
So, $f(c) < f(A) < f(B)$.
Then, $C - A - B$.
But $C \in AB$, so either $C = A$, $C = B$, or $A - C - B$
or $A - B - C$.
This conflicts with $C - A - B$. by HW4.
Thus, we get a contradiction in case $I(A)$.
Case $I(A)$: Suppose $y_b < y_a$.
Then, $y_c - y_a = t(y_b - y_a) > 0$.
 $< b < 0$
So, $f(B) < f(A) < f(C)$.
Thus, $B - A - C$.

~

Ease 2: Suppose
$$l = Lm,d$$
.
If you do the same arguments
as case 1, but use the standard
ruler for Lm,d then you
will get a contradiction also.

Let
$$A = (d, y_n), B = (d, y_b), C = (d, y_c).$$

Then $C = A + t(B - A)$ becomes
 $(d, y_c) = (d, y_n + t(y_b - y_n))$ (*)
 $(d, y_c) = (d, y_n + t(y_b - y_n))$ (*)
 $(d, y_c) = (d, y_n + t(y_b - y_n))$ (*)
Let $f: L \rightarrow IR$ be the standard ruler
Where $f(d, y) = y.$
Case (i): Suppose $y_n < y_b$, that is $f(A) < f(B)$
Let $g: L \rightarrow IR$ be
given by $g(x) = f(x) - f(A).$
Ry topic 3, g is a ruler on $L.$
Ry topic 3, g is a ruler on $L.$
Ry topic 3, g is a ruler on $L.$
Ry topic 3, g is a ruler on $L.$
Since $g(A) = 0$ and $g(B) > 0$ we know
 $\overrightarrow{AB} = \{x \in L \mid g(x) \ge 0\}.$
 $(bpic 5 notes)$

Note that
$$g(d,y) = f(d,y) - f(d,y_n)$$

 $= y - y_n$
Thus applying g to $(*)$ gives
 $y_c - y_n = (y_n + t(y_b - y_n)) - y_n$.
 $g(c)$
 $g(c) = t \cdot g(B) \ge 0$.
 $g(c) = t \cdot g(C) = 0$.
 $g(c) = t - g$

Also,
$$g(A) = f(A) - f(A) = 0$$

and $g(B) = f(A) - f(B) > 0$
Since $g(A) = 0$ and $g(B) > 0$ we know
 $\overrightarrow{AB} = \{ x \in L \mid g(x) \ge 0 \}$.
(hypic 5 notes)
 $g(x) = f(d, y) = f(d, y)$
 $= y_a - y$
Thus applying g to $(*)$ gives
 $y_a - y_c = y_a - (y_a + t(y_b - y_a))$
Therefore, $y_a - y_c = t(y_a - y_b)$
So, $g(c) = t \cdot g(B) \ge 0$.
Thus, $C \in \overrightarrow{AB}$.

Therefore, care (
$$ii$$
) is done.
In both cures we get $C \in \overrightarrow{AB}$.
Thus, $S \subseteq \overrightarrow{AB}$.

ト

Therefore,
$$S = AB$$
.



(ii) Let $A = (x_1, y_1)$ and $B = (x_2, y_2)$ with $X_1 < X_2$. Suppore A and B both lie on chr. Suppose that C=(x,y) lies on Lr and that $X_1 < X < X_2$. We must show that CEAB. Since C+A and C+B this comes down to showing that A-C-B We Knuw A, B, C are distinct points all lying on clr. Let f: L, -> IR be the standard ruler given by $f(a,b) = ln(\frac{a-c+r}{b})$. Recall that f": R-> Lr is given by f'(t) = (c+rtanh(t), rsech(t))

Let
$$f(A] = t_1, f(B) = t_2, f(c) = t_2$$
.

Then

$$X_{1} = c + r \tanh(t_{1})$$

$$X_{2} = c + r \tanh(t_{2})$$

$$X_{2} = c + r \tanh(t_{2})$$

$$X = c + r \tanh(t_{2})$$

$$X = c + r \tanh(t_{2})$$

$$K = c +$$

We are given that
$$X_1 < X < X_2$$
.
Since tunh(s) is an increasing function
this implies that $c+rtanh(s)$ is
an increasing function,
So, $X_1 < X < X_2$ implies $t_1 < t < t_2$.
Thus, $f(A) < f(c) < f(B)$
Therefore $A - C - B$.